Overview course module "Stochastic Modelling"

I. Introduction

II. Actor-based models for network evolution

III. Co-evolution models for networks and behaviour

IV. **Exponential Random Graph Models**

   A. Definition

   B. Example: gossip about the boss

   C. Estimation / goodness of fit issues

A: Exponential Random Graph Models

- history: Frank & Strauss “Markov random graph” model (1986), Frank (1991) and Wasserman & Pattison (1996) generalised to exponential family distribution:

$$\Pr(X = x) \propto \exp\left(\sum_{k=1}^{K} \beta_k s_k(x)\right)$$

‘proportional to’
proportionality constant unknown and practically uncalculable(!)

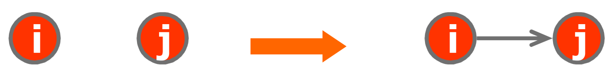
linear combination of network statistics
very flexible!

*Likelihood of macro structure **x** is explained by prevalence of micro structures **s(x)**, testable via parameters **β**.*

Some possible statistics

- tie count statistic

$$s_k(x) = \sum_{i,j=1}^{n} x_{ij}$$



- reciprocity statistic
  (for directed graphs)

$$s_k(x) = \sum_{i,j=1}^{n} x_{ij} x_{ji}$$

- transitive closure

$$s_k(x) = \sum_{i,j,k=1}^{n} x_{ij} x_{jk} x_{ik}$$

A positive parameter β attached to any of these statistics means: the right configuration is more likely than the left one.

Example

- Consider an ERG model for an undirected network with 3 parameters for these statistics:

(1) number of edges $s_{edges}(x) = \frac{1}{2} \sum_{i,j=1}^{n} x_{ij}$

$s_{2-stars}(x) = \frac{1}{2} \sum_{i,j,k=1}^{n} x_{ij} x_{ik}$ number of 2-stars (2)

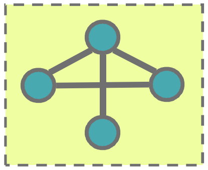
(3) number of triangles $s_{triangles}(x) = \frac{1}{6} \sum_{i,j,k=1}^{n} x_{ij} x_{jk} x_{ik}$

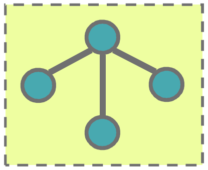
- Then the 3-parameter ERG distribution function is this one:

$$\Pr(X = x) \propto \exp(\beta_{edges} \times s_{edges}(x) + \beta_{2-stars} \times s_{2-stars}(x) + \beta_{triangles} \times s_{triangles}(x))$$

Example

› ...and consider the following two 4-node-networks & their statistics:

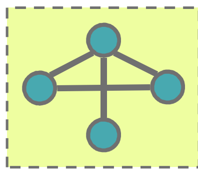




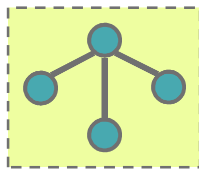
	x^a	x^b
s_{edges}	4	3
$s_{2\text{-}stars}$	5	3
$s_{triangles}$	1	0

Example

› Probabilities are (only) given up to a proportionality factor:



x^a

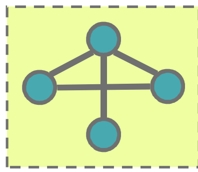


x^b

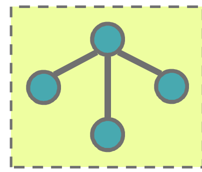
$$\Pr(x^a) \propto \exp(4\times\beta_{edges} + 5\times\beta_{2\text{-}stars} + 1\times\beta_{triangles})$$

$$\Pr(x^b) \propto \exp(3\times\beta_{edges} + 3\times\beta_{2\text{-}stars})$$

…but the ratio of probabilities can be calculated!



x^a



x^b

$$\frac{\Pr(x^a)}{\Pr(x^b)} = \frac{\exp(4\beta_{edges} + 5\beta_{2-stars} + \beta_{triangles})}{\exp(3\beta_{edges} + 3\beta_{2-stars})}$$

$$= \exp\big((4-3)\beta_{edges} + (5-3)\beta_{2-stars} + \beta_{triangles}\big)$$

$$= \exp(\beta_{edges} + 2\beta_{2-stars} + \beta_{triangles})$$

Local characterisation of ERGMs

- ERG distribution is considered as a collection of local conditional tie probabilities

$$\frac{\Pr(x^a)}{\Pr(x^b)} = \exp\left(\sum_{k=1}^{K} \beta_k \left(s_k(x^a) - s_k(x^b)\right)\right)$$

compared are a network x^a
and a "neighbouring network" x^b

model parameters

differences in model statistics between x^a and x^b

What this equation refers to is the (conditional) probability of the one tie by which they differ!

Recall similar formula for dynamic networks

- Local characterisation of ERG models:

$$\frac{\Pr(x^a)}{\Pr(x^b)} = \exp\left(\sum_{k=1}^{K} \beta_k \left(s_k(x^a) - s_k(x^b)\right)\right)$$

- Local characterisation for actor-based network evolution models:

$$\frac{\Pr(x^c \rightarrow_i x^a)}{\Pr(x^c \rightarrow_i x^b)} = \exp\left(\sum_{k=1}^{K} \beta_k \left(s_{ik}(x^a) - s_{ik}(x^b)\right)\right)$$

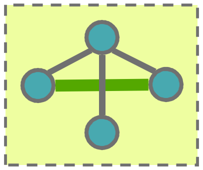
compared are two moves ('micro steps') made by actor ***i*** from a network $\boldsymbol{x^c}$ to two "neighbouring networks" $\boldsymbol{x^a}$ and $\boldsymbol{x^b}$

model parameters

difference in model statistics of actor ***i*** between the two compared moves

› …so how do these ‘conditional odds’ for the middle tie to exist (vs. not to exist) look like in applications?

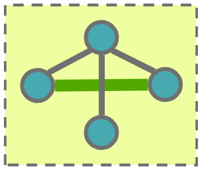
- Suppose, in a (larger) trade network, estimation gave:
  - redundant ties are avoided: $\beta_{triangles} = -0.4$
  - positive degree variance: $\beta_{2\text{-}stars} = 0.1$
  - low density: $\beta_{edges} = -1.5$



- Then the equation becomes:

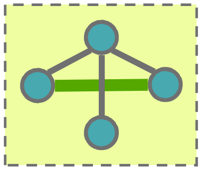
$$\frac{\Pr(\text{middle tie})}{\Pr(\text{no middle tie})} = \exp(\beta_{edges} + 2\beta_{2-stars} + \beta_{triangles})$$
$$= \exp(-1.5 + 2 \times 0.1 - 0.4)$$
$$= \exp(-1.7)$$
$$\approx 0.183$$
$$\approx 1/5.5$$

i.e., the middle tie is about
5.5 times as likely NOT to exist
as to exist [given the rest of the network].



› Same model in a different application:

- Suppose, in a (larger) friendship network, estimates were:
  - closure: $\beta_{triangles} = 0.4$
  - pos. degree variance: $\beta_{2\text{-}stars} = 0.1$
  - low density: $\beta_{edges} = -1.5$

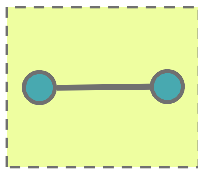


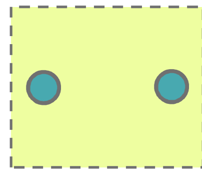
- Then the equation becomes:

$$\frac{\Pr(\text{middle tie})}{\Pr(\text{no middle tie})} = \exp(-0.9) \approx 0.407 \approx 1/2.5$$

- Still two½ times as likely NOT to exist – but always keep in mind that *all* ties are random…

...random ties are benchmark for evaluating such conditional odds! Hence, consider these “networks” as baseline comparison:





$$\frac{\Pr(\text{tie})}{\Pr(\text{no tie})} = \exp(\beta_{edges}) = \exp(-1.5) \approx 0.223 \approx 1/4.5$$

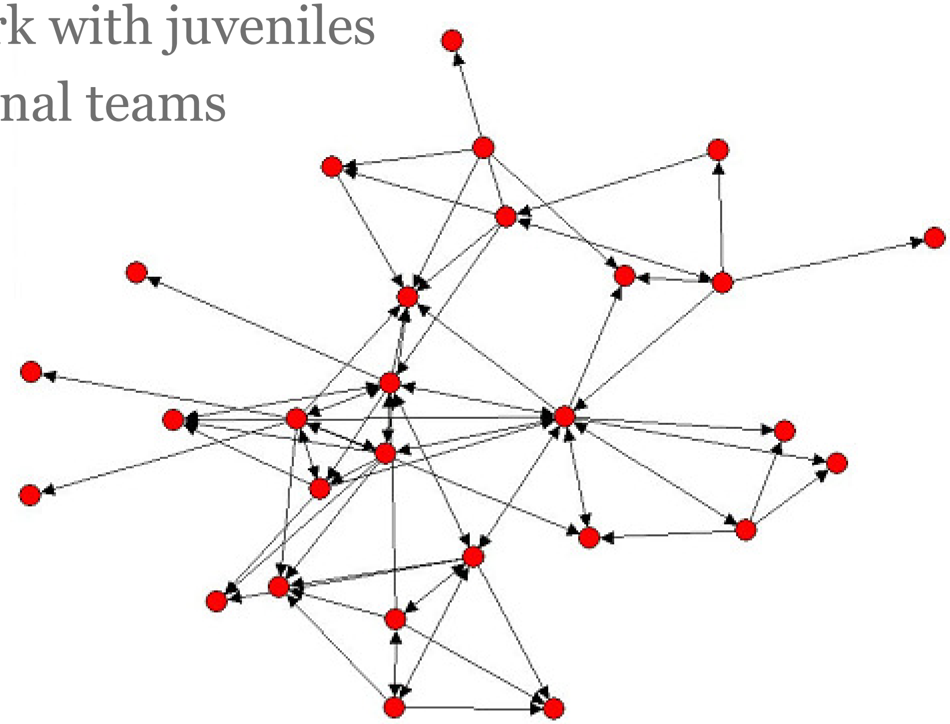
- In the *friendship context* (odds=2.5), the tie in question is *relatively likely* to be present,
- In the *trade context* (odds=5.5), it is *relatively unlikely* to be present, given the network’s density.

Exponential Random Graph Models

- high flexibility due to the many possibilities of choosing statistics & controlling effects for each other **+**
- can express small worlds (see Robins et al. 2005) but also other network types
- network probability model: not tied to any particular algorithm**+**
- problems: estimation, model specification, interpretation **–**

B: Example: gossip about the boss

› **data from Lea Ellwardt (2008)**
  - organisation for social work with juveniles
  - one department with internal teams
  - 28 employees



› *gossip about the boss in this department was largely* ***negative***

Hypotheses:

1. closeness / proximity is associated with gossip
    - friendship / liking predicts gossip
    - communication frequency predicts gossip
    - team structure predicts gossip
    - gossip occurs in local clusters
2. information asymmetry is associated with gossip
    - dissimilarity in / lack of contacts with the boss predicts g'sp.
3. "negative attitude" is associated with gossip
    - distrust in management predicts gossip
    - disliking the boss predicts gossip
4. attitude similarity is associated with gossip

Results from an exponential random graph analysis:

Parameter	*Estimate*	*SE*	
NETWORK EFFECTS			
1. Reciprocity	0.9356	0.6254	
2. Transitive triplets	-0.1851	0.2873	… (1)
3. 3-cycles	-0.1362	0.3619	… (1)
4. Alternating out-k-stars	**0.8823***	**0.3179**	← – – – Unpredicted: heterogeneity of outdegrees
5. Alternating in-k-stars	-0.2607	0.3701	
6. Alternating k-triangles	**0.9762***	**0.4834**	✓ (1)
7. Team membership	**0.7866***	**0.2835**	✓ (1)
8. Communication ego-alter (symmetric)	**0.9111***	**0.3254**	✓ (1)
9. Liking of alter by ego (out-degree)	**1.8782***	**0.3305**	✓ (1)

Results from an exponential random graph analysis:

Parameter	*Estimate*	*SE*	
SENDER EFFECTS			
10. Trust in management	-0.1261	0.1337	… (3)
11. Liking of manager	**-0.5071***	**0.2289**	✓ (3)
12. Communication with manager	**-0.3239***	**0.0915**	✓ (2)
RECEIVER EFFECTS			
13. Trust in management	0.2187	0.1996	… (3)
14. Liking of manager	0.2244	0.3588	… (3)
15. Communication with manager	-0.0445	0.1411	… (2)

Results from an exponential random graph analysis:

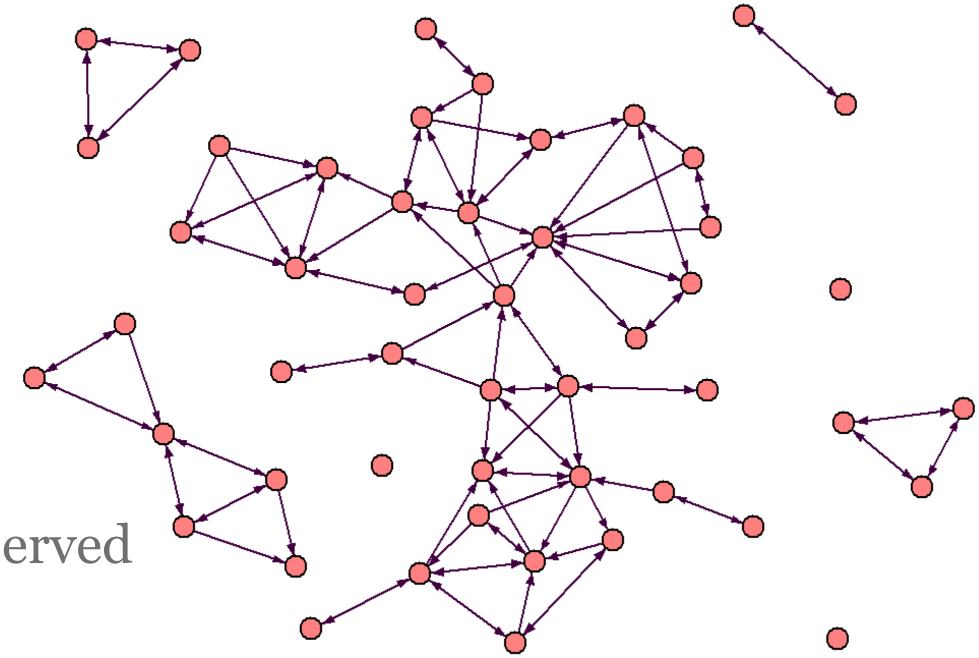
Parameter	*Estimate*	*SE*	
SIMILARITY EFFECTS			
16. Trust in management (similar)	**-1.3708***	**0.7921**	✓ (4)
17. Liking of manager (same)	0.3919	0.2878	… (4)
18. Communication with manager (similar)	-0.4359	0.6343	… (4)

* $p<0.05$ (one-sided test)

Conclusion: by and large, the analysis supports the hypotheses…

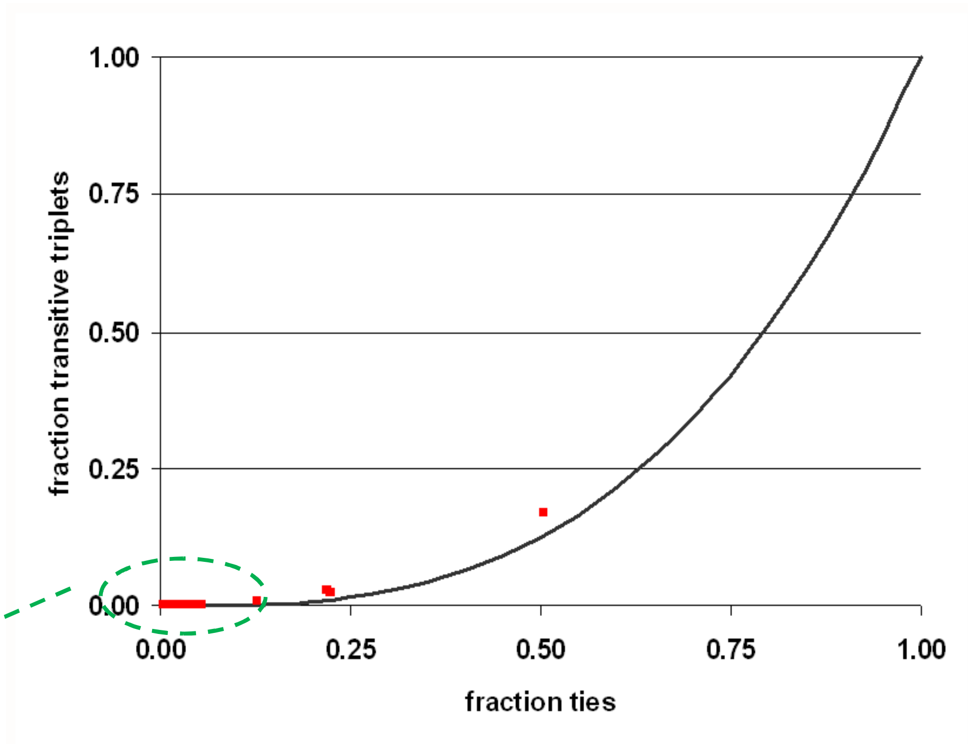
Example: 50 Scottish girls' nominated friendship

› **data from Michell & Amos (1997)**
  - 50 girls
  - Scotland 1995
  - 1st year secondary
  - 13 years old



› ***triangulation*** can be observed

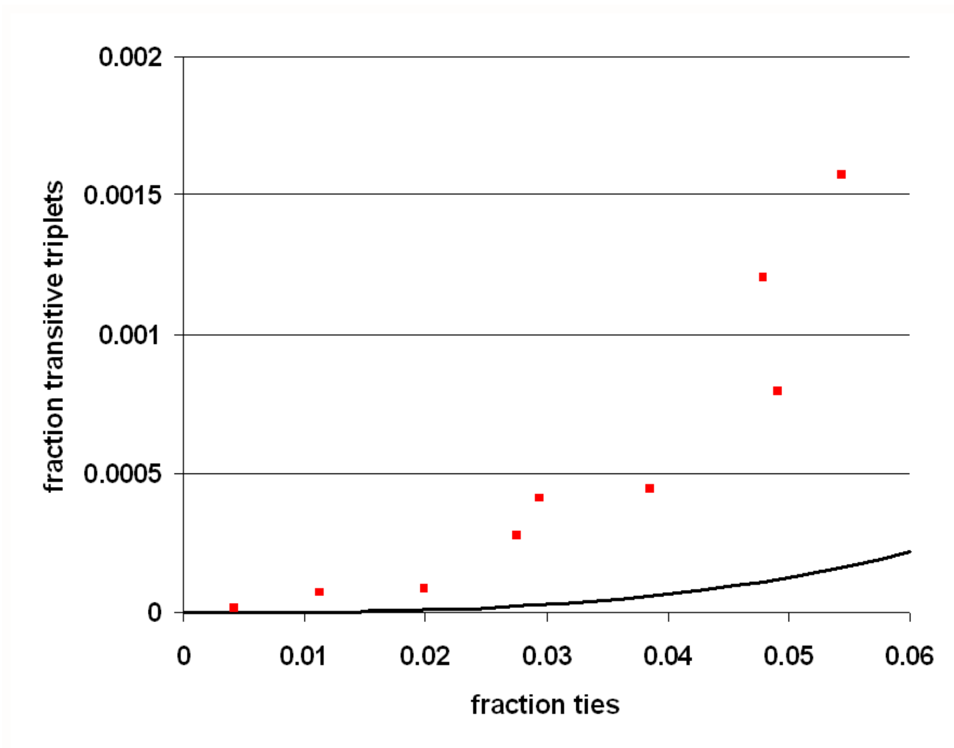
› triangulation general: typically more triangles in data than expected under random tie formation…



zooming in on the next slide …

Red markers stand for empirically observed data sets, black line for expectations under randomness.

› Red markers stand for observed values in several empirical data sets; the black line indicates expectations under randomness.

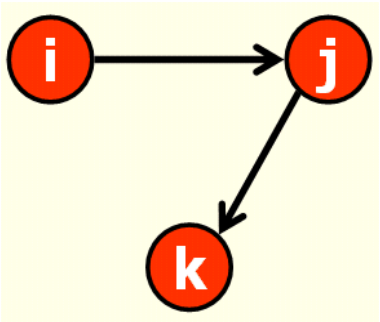


Red markers lie way above the random expectation.

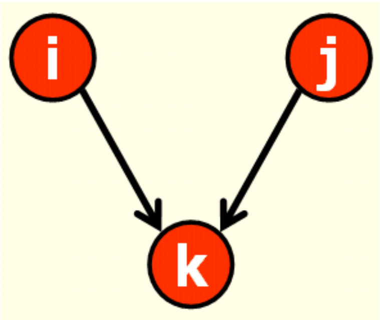
So: include triangulation in your model for this type of data!

First try: estimate a simple model with triangulation / transitive closure tendencies

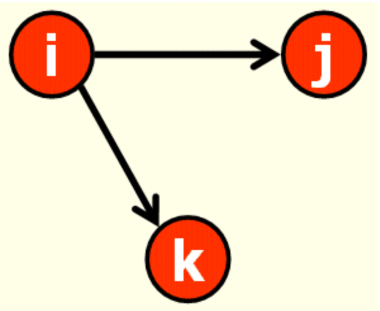
› include *precursor configurations* in attempt to arrive at a meaningful model (Snijders 2002)



2-path



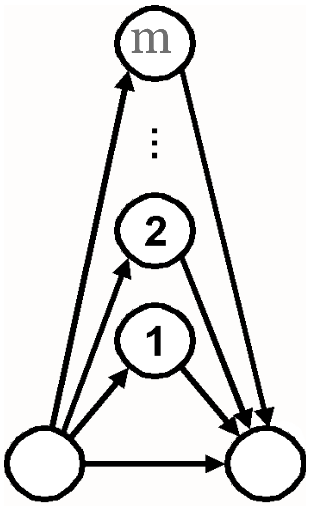
in-2-star



out-2-star

```
Good convergence is indicated by the t-statistics being close to zero.
One or more of the t-statistics are rather large.
Convergence of the algorithm is doubtful.

@2
Estimation results.
-------------------

Regular end of estimation algorithm. Total of 2092 iteration steps.

@3
Estimates and standard errors

 1. u: number of arcs                             4.9212  (   1.7544)
 2. u: reciprocity                                3.9062  (   0.3581)
 3. u: in-2-stars                                -1.2377  (   0.3367)
 4. u: 2-paths                                   -0.8630  (   0.1349)
 5. u: transitive triplets                      -12.7060  (  99.0000)
```

Second try: estimate another model with 'new specification' (Snijders et al. 2006)



$$s_k(x) = \sum_{m=1}^{n-2} (-1)^{m+1} \frac{\tau_m}{2^{m-1}}$$

"alternating m-triangles statistic"

τ_m = number of m-triangles (configurations on the left)

```
Good convergence is indicated by the t-statistics being close to zero.

@2
Estimation results.
-------------------

Regular end of estimation algorithm.
Total of 2015 iteration steps.

@3
Estimates and standard errors

 1. u: number of arcs                          -3.2498  (   0.3456)
 2. u: reciprocity                              4.4430  (   0.3867)
 3. u: alternating m-triangles, parameter 2     0.6734  (   0.0758)
```

Why does estimation seem so difficult for some models?
Why so complex statistics like alternating m-triangles?

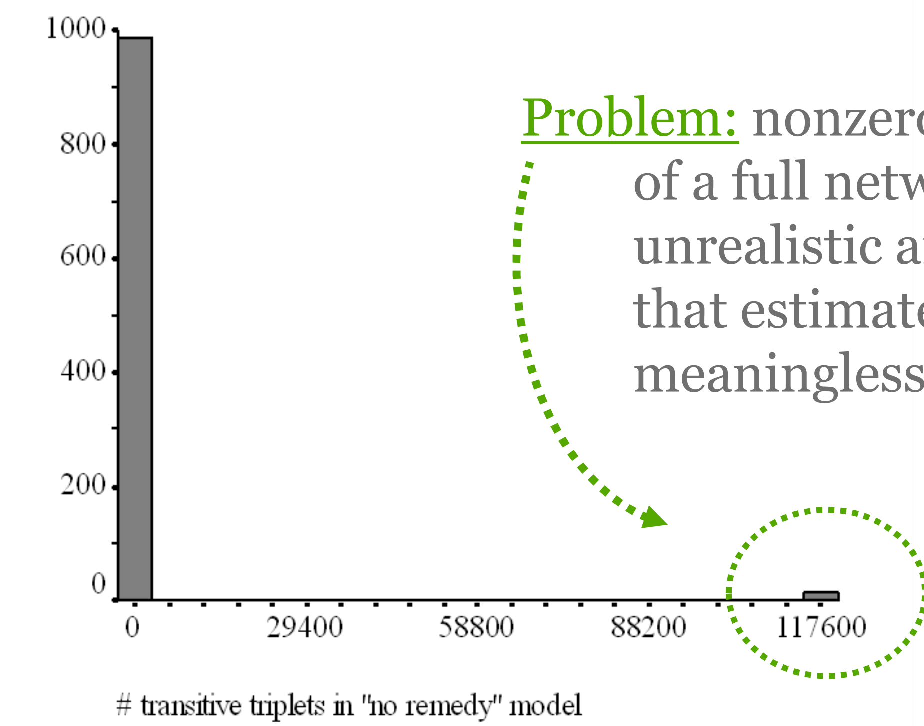
C: Estimation of ERGMs

› ***estimation is based on simulation:***

1. assume starting values for parameters
2. simulate networks that are drawn from this parametrisation's ERG distribution
3. compare them to actually observed (and to-be-modelled) network
4. adjust parameters in order to get **better fit** [on desired dimensions]
5. repeat from 2. until parameter values **converge**

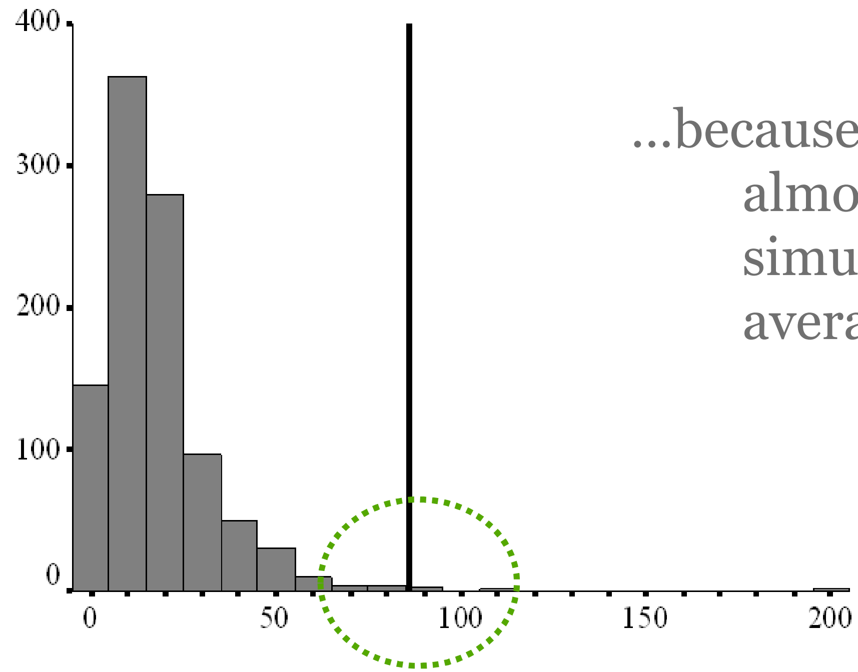
› this unfortunately does not always work,
and notably triangulation effects cause problems...

› see here the distribution of the number of transitive triplets over simulated networks from a **model like the first** we estimated...



Problem: nonzero probability of a full network! This is unrealistic and implies that estimates are meaningless...

› ...zooming in on the 'low end' (i.e., realistic networks; vertical line = observed value):



same graph with emphasis on the 'empty end'

...because observed value is almost never assumed by simulations – only "on average"!

To keep in mind about ERGMs:

- Visualise ERG models as probability distributions on a (huge) space of all possible network,
- one observed network is modelled as drawn from that distribution.
- Model parameters β are
  - attached to network statistics $\boldsymbol{s}$,
  - these statistics in general correspond to subgraph counts (local patterns, 'motifs'),
  - the parameters describe the relative prevalence of the corresponding subgraph in the total graph.
- Interpretation of parameters needs to take into account other parameters (notably edges / arcs).

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