Endogenous Network Dynamics

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Abstract

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the with whom) and the strategy (the how) of interactions change. Our objectives here are to model the structure and strategy of interactions prevailing at any point in time as a directed network and to address the following open question in the theory of social and economic network formation: given the rules of network and coalition formation, the preferences of individuals over networks, the strategic behavior of coalitions in forming networks, and the trembles of nature, what network and coalitional dynamics are likely to emergence and persist. Our main contributions are (i) to formulate the problem of network and coalition formation as a dynamic, stochastic game, (ii) to show that this game possesses a stationary correlated equilibrium (in network and coalition formation strategies), (iii) to show that, together with the trembles of nature, this stationary correlated equilibrium determines an equilibrium Markov process of network and coalition formation which respects the rules of network and coalition formation and the preferences of individuals, and (iv) to show that, although uncountably many networks may form, this endogenous process of network and coalition formation possesses a nonempty *finite* set of ergodic measures and generates a *finite*, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction. Moreover, we extend to the setting of endogenous Markov dynamics the notions of pairwise stability (Jackson-Wolinsky, 1996), strong stability (Jackson-van den Nouweland, 2005), and Nash stability (Bala-Goyal, 2000), and we show that in order for any network-coalition pair to be stable (pairwise, strong, or Nash) it is necessary and sufficient that the pair reside in one of finitely many basins of attraction - and hence reside in the support of an ergodic measure. The results we obtain here for endogenous network dynamics and stochastic basins of attraction are the dynamic analogs of our earlier results on endogenous network formation and strategic basins of attraction in static, abstract games of network formation (Page and Wooders, 2008), and build on the seminal contributions of Jackson and Watts (2002), Konishi and Ray (2003), and Dutta, Ghosal, and Ray (2005).

1 Introduction

1.1 Overview

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the with whom) and the strategy (the how) of interactions change. Our objectives here are to model the structure *and* strategy of interactions prevailing at any point in time as a directed network and to address the following open question in the theory of social and economic network formation: given the rules of network formation, the preferences of individuals over networks, the strategic behavior of coalitions in forming networks, and the trembles of nature, what *network and coalitional dynamics* are likely to emergence and persist. Thus, we propose to study the emergence of endogenous network and coalitional dynamics from strategic behavior and the randomness in nature.

Our main contributions are (i) to formulate the problem of network formation as a dynamic, stochastic game, (ii) to show that this game possesses an equilibrium in stationary correlated network and coalition formation strategies, (iii) to show that, together with the trembles of nature, these equilibrium strategies (stationary correlated equilibrium network and coalition formation strategies) determine an equilibrium Markov process of network and coalition formation which respects the rules of network formation and the preferences of individuals, and (iv) to show that, although uncountably many networks may form, the equilibrium Markov process of network and coalition formation possesses only a *finite* number of ergodic probability measures and generates only *finite* number nonempty subsets of networks and coalitions, each constituting a *stochastic basin of attraction*.

In our prior work on static abstract games of network formation (Page and Wooders, 2007, denoted PW07), we have shown that, given the rules of network formation and the preferences of individuals, these games possess *strategic basins of attraction* and these contain all networks that are likely to emerge and persist as the game unfolds. Moreover, we have shown that when any one of these strategic basins contains only one network, then the game possesses a network (i.e., the single network contained in the singleton basin) that is stable against all coalitional network deviation strategies - and thus the game has a nonempty *path dominance core*. Finally, we have shown in PW07 that depending on how we specialize the rules of network formation and the dominance relation over networks, any network contained in the path dominance core is pairwise stable (Jackson-Wolinsky, 1996), strongly stable (Jackson-van den Nouweland, 2005), Nash (Bala-Goyal, 200), or consistent (Chwe, 1994).

We show here that there are many parallels between the static abstract game formulation and our prior results for static games and the results we obtain here for our Markov dynamic game formulation. This is suggested already by the seminal paper by Jackson and Watts (2002) on the evolution of networks. Jackson and Watts present a basic theory (and to our knowledge the first theory) of stochastic dynamic network formation over a finite set of linking networks governed by Markov chain generated by myopic players (following the Jackson-Wolinsky rules of network formation) and the trembles of nature. Their model builds on the earlier, nonstochastic model of dynamic network formation due to Watts (2001) - as far as we know, the first models of network dynamics are Watts (2001) and Skyrms and Pemantle (2000). By considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain, they show that any pairwise stable network is necessarily contained in the support of an invariant measure - that is, in the support of a probability that places all its support on sets of networks likely to form in the long run. We show here that similar conclusions can be reached for directed networks with many arc types governed by arbitrary network formation rules.

In a general Markov game setting, with farsighted players, what precisely does it mean for a network to be pairwise stable - or stable in any sense? For example, if the state space of networks is large, then the endogenous Markov process of network formation is likely to have many invariant measures - and in fact many ergodic probability measures (i.e., measures that place all their probability mass on a single absorbing set). Which absorbing set contains networks stable in the sense of pairwise stability, or strong stability, or Nash stability? These are some of the questions we answer here in our study of endogenous network dynamics.

We conjecture that in any reasonable dynamic, stochastic model of network formation the endogenously determined Markov process of network and coalition formation will possess ergodic probability measures and generate basins of attraction. We show here that in fact the endogenous Markov process possesses only finitely many ergodic measures and basins of attraction. This endogenous finiteness property of equilibrium has serious implications for empirical work on networks. In particular, since nature does not afford the empirical observer multiple observations across states but rather only multiple observations across time, the fact that only finitely many long run equilibrium sets are possible and more importantly, the fact that on these sets (i.e., on these basins of attraction) state averages are equal to time averages gives meaning and significance to time series observations which seek to infer the long run equilibrium network. Moreover, to the extent that networks can truly represent various social and economic interactions, our understanding of how and why the network formation process moves toward or away from any particular basin can potentially shed new light on the persistence or transience of many social and economic conditions. For example, how and why does a particular path of entrepreneurial and scientific interactions carry an economy beyond a tipping point and onto a path of economic growth driven by a particular industry - and why might it fail to do so? How and why does a particular path of product line-nonlinear pricing schedule configurations lead a strategically competitive industry to become more concentrated - or fade? These are some of the applied questions which hopefully can be addressed using a model of endogenous network dynamics

1.2 Endogenous Network Dynamics

Our approach to endogenous dynamics is motivated by the observation that the stochastic process governing network and coalition formation through time is determined not only by nature's randomness (or nature's trembles) through time - as envisioned in random graph theoretic approaches - *but also* by the strategic behavior of individuals and coalitions through time in attempting to influence the networks and coalitions that emerge under the prevailing rules of network formation and the trembles of nature. Thus, here we will develop a theory of endogenous network and coalitional dynamics that brings together elements of random graph theory and game theory in a dynamic stochastic game model of network and coalition formation. While dynamic stochastic games have been used elsewhere in economics (see, for example, Amir (1991, 1996), Amir and Lambson (2003), and Chakrabarti (1999, 2008), Duffie, Geanakoplos, Mas-Colell, and McLennan (1994), Mertens and Parthasarathy (1987, 1991), Nowak (2003, 2007)), their application to the analysis of the evolution of social and economic networks is relatively new.

Our plan of analysis has two parts. In part (1) we will construct our dynamic game model of network *and* coalition formation, and then show that this game has an equilibrium in stationary correlated stationary strategies. Our model has six primitives consisting of the following: (i) a feasible set of directed networks representing all possible configurations of social or economic interactions, (ii) a feasible set of coalitions allowed to form under the rules of network formation for the purpose of proposing alternative networks, (iii) a state space consisting of feasible network-coalition pairs, (iv) a set of players and player constraint correspondences specifying for each player and in each state the set of feasible alternative networks and coalitions that a player can propose under the rules of network formation as a member of the current or status quo coalition - and as a nonmember, (v) a set of player discount rates and payoff functions defined on the graph of players' product constraint correspondence, and (vi) a stochastic law of motion. This stochastic law of motion represents nature and specifies the probability with which each possible new status quo network-coalition (i.e., new state) might emerge as a function of the status quo network-coalition pair (i.e., the current state) and the profile of player-proposed new status quo network-coalition pairs (i.e., the current action profile). Using these primitives, we will construct a discounted stochastic game model of network formation, and then show that this game possesses a stationary correlated equilibrium in network-coalition proposal strategies.

Finally, in part (1) we will show that, together with the stochastic law of motion, these stationary correlated equilibrium strategies determine an equilibrium Markov process of network and coalition formation. More importantly, we will be able to conclude via classical results due to Blackwell (1965) (also, Himmelberg, Parthasarathy, and vanVleck (1976)), Nowak and Raghavan (1992), and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)) that these stationary correlated equilibrium strategies are optimal against player defections to *any other history-dependent* network-coalition proposal strategies - thus showing that our decision to focus on stationary correlated strategies is well-founded.

In part (2), we will analyze the stability properties the endogenous Markov process of network and coalition formation. In particular, using methods of stability analysis essentially due to Nummelin (1984) and Meyn and Tweedie (1993) - and based on the profound work of Doeblin (1937, 1940) - we will show that the equilibrium Markov process of network and coalition formation possesses ergodic probability measures and generates basins of attraction. We will then study in some detail the number and structure of these basins of attraction as well as the structure of set of invariant probability measures. More importantly we will show that the equilibrium process possesses only finitely many ergodic measures and basins of attraction. Finally, in part (2), we will extend the definitions of pairwise stable, strongly stable, Nash, and consistent networks to the dynamic Markov setting of the model and show that these various types of stable networks can be found only in the basins of attraction generated under the appropriate specification of the rules of network formation and feasible coalitions.

1.3 Related Literature

To our knowledge, the first paper to study endogenous dynamics in a related model is the paper by Konishi and Ray (2003) on dynamic coalition formation. The primitives of their model consist of (i) a finite set of outcomes (possibly a finite set of networks), (ii) a set of coalitional constraint correspondences specifying for each coalition and each status quo outcome, the set of new outcomes a coalition might bring about if allowed to do so, and (iii) a discount rate and set of player payoff functions defined on the set of all outcomes. Konishi and Ray show that their model possesses an equilibrium process of coalition formation, that is, a stochastic law of motion governing movement from one outcome to another such that (a) if a move from one outcome to another takes place with positive probability, then for some coalition this move makes sense in that no coalition member is made worse off by the move and no further move makes all coalition members better off, and (b) if for a given outcome there is another outcome making all members of some coalition better off and no further outcome makes this coalition even better off, then a move to another outcome takes place with probability 1 (i.e., the probability of standing still at the given outcome is zero). The notion of a player being better off is reckoned in terms of a player's valuation function implied by the maximization of the expected discounted stream of payoffs with respect to the stochastic law of motion. Stated loosely, then, Konishi and Ray show that for their model there is a law of motion which generates coalitionally Pareto improving moves from one outcome to another (i.e., in our case it would be from one network to another).

Our model differs from the model of Konishi and Ray in several respects. First, in our model movements from one network (outcome) to another are largely determined by the strategic behavior of individuals within feasible coalitions. In Konishi and Ray, coalitions are passive and strategic behavior plays no part in determining the movement from one outcome to another. They simply show that there model is consistent with there being a law of motion which moves the outcome along in a coalitionally Pareto improving way. In this sense - i.e., in the sense that movement is nonstrategic - their model is more closely related to random graph theoretic models of network dynamics. In our model, equilibrium strategic behavior, together with natures trembles, are central to determining equilibrium network dynamics.

Second, whereas Konishi and Ray, for technical reasons, restrict attention to a finite set of outcomes (in our model, a finite set of networks), we allow for uncountably many networks - this to allow for consideration of networks with a large number of nodes or networks with uncountably many arc types. This generalization is more than a technical nicety. In order to capture the myriad and potentially complex nature of interactions between players (say for example in a stock market or in a contracting game with multiple principals and multiple agents) we must allow there to be uncountably many possible types of interactions. In our model the set of potential interactions are represented by a set of arc types with each arc type (or arc label) representing a particular type of interaction (or connection) between nodes in a directed network. Thus, because we allow for uncountably many arc types in describing the possibly finite number of interactions between nodes, in our model there are uncountably many possible networks (or outcomes in the language of Konishi and Ray). Moreover, in order to model large networks (i.e., networks with many nodes), in our model we can allow there to be infinitely many nodes - although here we focus exclusively on the finite nodes case. Third, while Konishi and Ray restrict attention at the outset to Markov laws of motion, we will show that our strategically determined equilibrium Markov process of network and coalition formation is robust against all possible alternative dynamics induced by history-dependent types of strategic behavior. Thus, at least for the class of Konishi-Ray types of models, we will show that Markov laws of motion are stable and robust with respect to other forms of history-dependent laws of motion.[1]

Finally, whereas Konishi and Ray focus on the existence of an equilibrium process of coalition formation, here we will not only establish the existence of a strategically determined equilibrium process of network and coalition formation, but also we will show that this process possesses a nonempty set of ergodic measures and generates basins of attraction.

Dutta, Ghosal, and Ray (2005) extend the Konishi-Ray type model to consider a particular form of strategic behavior (i.e., strategic behavior governed by a particular set of network formation rules) in a dynamic game of network formation over a finite set of undirected linking networks (rather than directed networks). They show that their model has a Nash equilibrium and identify conditions under which efficiency can be sustained in equilibrium - thus, continuing in a dynamic setting the seminal work of Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) on equilibrium and efficiency. Here our focus is on equilibrium and stability rather than equilibrium and efficiency and our analysis is carried out in a dynamic, stochastic game model of network and coalition formation, admitting all forms of network formation rules, over an uncountable set of directed networks. While Dutta, Ghosal, and Ray restrict attention to Markov strategies and show that there is an equilibrium in this class of network formation strategies, here, we show that there is an equilibrium in the class of all stationary correlated network and coalition proposal strategies and that this type of equilibrium is optimal relative to the class of *all* history-dependent net-

[1] By a Markov law of motion we mean a stochastic law of motion where probabilistic movements from one outcome or network to another depend only on the current outcome rather than on some history of outcomes.

work formation strategies. Moreover, as mentioned above, we show that in general, the resulting equilibrium Markov network and coalitional dynamics possess ergodic measures and generate network and coalitional basin of attraction.

We view the starting point of our research to be the pioneering work of Jackson and Watts (2002) already discussed briefly above. Our model of endogenous network and coalitional dynamics extends their work on stochastic network dynamics in several respects. First, in our model players behave farsightedly in attempting to influence the path of network and coalition formation - farsighted in the sense of dynamic programing (e.g.,Dutta, Ghosal, and Ray (2005))[2]. Moreover, in our model the game is played over a (possibly) uncountable collection of directed networks under general rules of network formation which include not only the Jackson-Wolinsky rules, but also other more complex rules. In our model the law of motion is such that the trembles of nature are Markovian rather than i.i.d. as in Jackson and Watt, and are functions of the current state and the current profile of network and coalition proposals by players. Extending the notion of pairwise stability to a dynamic setting, one of the benchmarks for our research is to show that in a Markov model of network and coalition formation, if a network is dynamically pairwise stable, then it must be contained in one of finitely many basins of attraction, and therefore, contained in the support of an ergodic probability measure.

[2] See Chwe (1994), Page, Wooders, and Kamat (2005), and Page and Wooders (2007) for notions of farsighted behavior in static, abstract games.

2 Primitives

2.1 The Space of Directed Networks

We begin by giving the formal definition of a directed network. Let N be a finite set of nodes with typical element denoted by i and let A be a compact metric space of arcs with typical element denoted by a. Denote by d_A the metric on A and by d_N the *discrete metric* on N.[3] Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.

Definition 1 (*Directed Networks*)

Given node set N and arc set A, a directed network, G, is a nonempty, closed subset of $A \times (N \times N)$. The collection of all directed networks is denoted by $P_f(A \times (N \times N))$.

A directed network $G \in P_f(A \times (N \times N))$ thus consists of a set of ordered pairs of the form $(a, (i, i'))$ where a is an arc type or an arc label and (i, i') is an ordered pair of nodes. We shall refer to any pair $(a, (i, i')) \in G$ as a *connection* in network G. Thus, a network G is a closed set of connections specifying how the nodes in N are connected by the arcs in A. In a directed network order matters. In particular, $(a, (i, i')) \in G$ means that nodes i and i' are connected by a type a arc *from* node i *to* node i'.

Note that under our definition of a directed network, loops are allowed - that is, we allow an arc to go from a given node back to that given node.[4] Finally, note that under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow an arc a to go from a node i to a node i' multiple times.

The following notation is useful in describing networks. Given directed network $G \in P_f(A \times (N \times N))$, let

$$G(a) := \left\{ (i, i') \in N \times N : (a, (i, i')) \in G \right\},$$

$$G^+(i) := \left\{ a \in A : (a, (i, i')) \in G \text{ for some } i' \in N \right\},$$

and

$$G^-(i') := \left\{ a \in A : (a, (i, i')) \in G \text{ for some } i \in N \right\}.$$

Thus, in network G,

[3] The discrete metric d_N is given by

$$d_N(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

[4] By allowing loops we are able to represent a network having no connections between distinct nodes as a network consisting entirely of loops at each node.

$G(a)$ is the *set of node pairs* connected by arc a,
$G^+(i)$ is the *set of arcs* leaving node i, and
$G^-(i')$ is the *set of arcs* entering node i'.

If for some arc $a \in A$, $G(a)$ is empty, then arc a is not used in network G. Also, if for some node $i \in N$, $G^+(i) \cup G^-(i)$ is empty, then node i is said to be isolated.

Because $A\times(N\times N)$ is a compact metric space, the set of networks $P_f(A\times(N\times N))$ equipped with the Hausdorff metric h is a compact metric space (see Aliprantis and Border (1999), sections 3.14-3.16). Formally, the Hausdorff metric is defined as follows: First, let the distance between connection $(a,(i_0,i_1)) \in A \times (N \times N)$ and network $G \in P_f(A \times (N \times N))$ be given by

$$d((a,(i_0,i_1)),G) := \inf_{(a',(i_0',i_1'))\in G} d\left((a,(i_0,i_1)),(a',(i_0',i_1'))\right),$$

where

$$d\left((a,(i_0,i_1)),(a',(i_0',i_1'))\right) := d_A(a,a') + d_N(i_0,i_0') + d_N(i_1,i_1')$$

is the metric on $A \times (N \times N)$. Given this distance measure between connections and networks, the Hausdorff metric h is then defined as

$$h(G,G') := \max\left\{\sup_{(a,(i_0,i_1))\in G} d((a,(i_0,i_1)),G'), \sup_{(a',(i_0',i_1'))\in G'} d((a',(i_0',i_1')),G)\right\},$$

for G and G' in $P_f(A \times (N \times N))$.[5]

Given the nature of the discrete metric on the set of nodes, it is easy to see that if the Hausdorff distance between networks G and G' is less than $\varepsilon \in (0,1)$, that is, if networks G and G' are within ε distance for $\varepsilon < 1$, then the same set of nodes are involved in connections in both networks and the networks differ only in the way these nodes are connected (i.e., in the types of arcs used in making the connections). Thus if $h(G,G') < \varepsilon < 1$, then

$$(a,(i,i')) \in G \text{ if and only if } (a',(i,i')) \in G'$$

for arcs a and a' with $d_A(a,a') < \varepsilon$.

Convergence in the space of directed networks $(P_f(A\times(N\times N)),h)$ can be characterized via the notions of limes inferior and limes superior. Let $\{G^n\}_n$ be a sequence of directed networks. The limes inferior of this sequence, denoted by $Li(G^n)$, is defined as follows: connection $(a,(i,i')) \in Li(G^n)$ if and only if there is a sequence of connections $\{(a^n,(i^n,i'^n))\}_n$ converging to $(a,(i,i'))$ (i.e., $(a^n,(i^n,i'^n)) \xrightarrow{d} (a,(i,i'))$) where for each n connection $(a^n,(i^n,i'^n))$ is contained in network G^n. The limes superior, denoted by $Ls(G^n)$, is defined as follows: connection $(a,(i,i')) \in Ls(G^n)$ if and only if there is a subsequence of connections $\{(a^{n_k},(i^{n_k},i'^{n_k}))\}_k$ converging to $(a,(i,i'))$

[5] It is important to note that because $A \times (N \times N)$ is compact, all metrics compatible with the product topology on $A\times(N\times N)$, generate the same Hausdorff metric topology on $P_f(A\times(N\times N))$ (see Theorem 3.77 in Aliprantis and Border (1999)).

(i.e., $(a^{n_k}, (i^{n_k}, i'^{n_k})) \xrightarrow{d} (a, (i, i')))$ where for each k connection $(a^{n_k}, (i^{n_k}, i'^{n_k}))$ is contained in network G^{n_k}. A directed network $G \in P_f(A \times (N \times N))$ is said to be the limit of networks $\{G^n\}_n$ if $Ls(G^n) = G = Li(G^n)$. Moreover, $Ls(G^n) = G = Li(G^n)$ if and only if $h(G^n, G) \to 0$ (i.e., the sequence of networks $\{G^n\}_n$ converges to network $G \in P_f(A \times (N \times N))$ under the Hausdorff metric h).[6]

In formulating our game of network and coalition formation, it will often be useful to restrict attention to a particular feasible subset of networks.

Definition 2 (*Feasible Networks*)

Given node set N and arc set A, a feasible set of networks is a nonempty, h-closed subset $\mathbb{G}$ of the collection of all directed networks $P_f(A \times (N \times N))$.

Example (A feasible set of networks): Suppose that the feasible set of networks $\mathbb{G}$ is given by

$$\mathbb{G} = \left\{ G \in P_f(A \times (N \times N)) : \left| G^+(i) \right| \leq c(i) \right\},$$

where $c(\cdot)$ is a nonnegative integer-valued function and $|G^+(i)|$ denotes the cardinality of the set of arcs $G^+(i)$ emanating from node i (i.e., the out degree of node i). Thus, in each network G contained in $\mathbb{G}$ there is *at most* $c(i)$ arcs emanating from node i. It is easy to show that $\mathbb{G}$ is an h-closed subset of $P_f(A \times (N \times N))$.

2.2 Players and Coalitions

In our game theoretic model of network and coalition formation we will make a distinction between the set of players (or decision makers) and the set of nodes. In particular, we will not assume that the set of players and the set of nodes are necessarily one and the same. For example, some nodes may be club locations while other nodes may be players who choose clubs.

Because changing one network to another network very often involves groups of players acting in concert, coalitions will play a central role in our model. Let D denote the set of players (a set not necessarily equal to N the set of nodes) with typical element denoted by d and let $P(D)$ denote the collection of all coalitions (i.e., nonempty subsets of D) with typical element denoted by S. We will assume that the set of players D has cardinality m (i.e., $|D| = m$). Depending on the rules of network formation, it will often be useful to restrict attention to a particular feasible subset of coalitions.

[6] Both $Li(G^n)$ and $Ls(G^n)$ are networks, that is, both $Li(G^n)$ and $Ls(G^n)$ are contained in $P_f(A \times (N \times N))$. Moreover, in general,

$$Li(G^n) \subseteq Ls(G^n).$$

Definition 3 (*Feasible Coalitions*)

Given player set D, a feasible set of coalitions is a nonempty subset $\mathcal{F}$ of the collection of all coalitions $P(D)$.

Example (A set of coalitions): Suppose that the feasible set of coalitions is given

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

Thus, all feasible coalitions consist of at most two players. The set $\mathcal{F}_2$ is, for example, the feasible set for the Jackson-Wolinsky rules

2.3 States, Actions, and Payoffs

We shall take as the state space the space $\Omega := (\mathbb{G}\times\mathcal{F})$ of all feasible network-coalition pairs. Each state in $(\mathbb{G}\times\mathcal{F})$ has the following interpretation: if (G, S) is the current state, then G is the current status quo network of social interactions and it is coalition S's turn to propose a new state - that is, to propose a new status quo network and a new coalition to propose the next network. Once we have gotten out of the way some technical issues, we will return to a discussion of how we model movement from one state to another in our game.

Equipping $\mathcal{F}$ with the discrete metric $d_{\mathcal{F}}$ (i.e., $d_{\mathcal{F}}(S', S) = 0$ if $S' = S$, $d_{\mathcal{F}}(S', S) = 1$ if $S' \neq S$), the state space $(\mathbb{G}\times\mathcal{F})$ is a compact metric space under the metric d_Ω given by

$$d_\Omega((G', S'), (G, S)) := h(G', G) + d_{\mathcal{F}}(S', S).$$

Letting $B(\Omega) := B(\mathbb{G} \times \mathcal{F})$ be the Borel σ-field generated by the metric d_Ω, we equip our state space $(\mathbb{G}\times\mathcal{F}, B(\mathbb{G} \times \mathcal{F}))$ with a probability measure

$$\mu = \nu \times \gamma$$

where the probability measure γ on coalitions is such that $\gamma(S) > 0$ for all $S \in \mathcal{F}$ and where the probability measure ν on networks is such that the, at most, countable set of networks constituting the set of all atoms of the dominating probability measure ν is given by

$$\mathbb{A}_\nu = \{G_{\alpha 1}, G_{\alpha 2}, \ldots\} = \{G_{\alpha k}\}_{k=1}^{\infty} \subset \mathbb{G}. \tag{1}$$

For all $G_{\alpha k} \in \mathbb{A}_\nu$, $\nu(\{G_{\alpha k}\}) > 0$ and for all networks $G \in \mathbb{G}\backslash\mathbb{A}_\nu$, $\nu(\{G\}) = 0$ (Parthasarathy (1967)). Thus, we have as our state space, the probability space

$$(\Omega, B(\Omega), \mu) = (\mathbb{G}\times\mathcal{F}, B(\mathbb{G} \times \mathcal{F}), \nu \times \gamma), \tag{2}$$

a compact metric space with metric $d_\Omega = h + d_{\mathcal{F}}$. Because $\mathbb{G}$ is a compact metric space, $B(\mathbb{G} \times \mathcal{F}) = B(\mathbb{G}) \times \mathbb{B}(\mathcal{F})$ and $\mathbb{B}(\mathcal{F})$ is the set of all subsets of $\mathcal{F}$ (including the empty set).

In our game each player's action takes the form of a recommendation or proposal. In particular, given current state $(G, S) \in \Omega$, each player $d \in D$ has available a nonempty, closed set of actions $\Phi_d(G, S) \subseteq \mathbb{G}\times\mathcal{F}$ - that is, a set of proposals - such that player d, if given the power, could implement any proposal in $\Phi_d(G,S)$. We will assume that

$$\Phi_d(G,S) = \Gamma_d(G,S) \times \Lambda_d(G,S),$$

where for each state (G,S), $\Gamma_d(G,S)$ is a subset of $\mathbb{G}$ and $\Lambda_d(G,S)$ is a subset of $\mathcal{F}$.

Thus, the collection of constraint mappings

$$\{\Phi_d(\cdot) : d \in D\}$$

specifies, for any current state (G,S), the set of network-coalition proposals available to each player. These mappings specify the rules of network formation by specifying for each player d, in each state of the game (G,S), the possible moves that player d could make if given the power to do so. With this in mind, we will assume that

A-1 (*continuity of the constraint mappings*)

for each player $d \in D$, the correspondence $\Phi_d(\cdot)$ is such that

(i) $\Phi_d(\cdot)$ has a closed graph,

$$Gr\Phi_d(\cdot) := \left\{ \left((G,S), (G',S') \right) \in (\mathbb{G}\times\mathcal{F}) \times (\mathbb{G}\times\mathcal{F}) : (G',S') \in \Phi_d(G,S) \right\}, \tag{3}$$

(ii) for each state $(G,S) \in (\mathbb{G}\times\mathcal{F})$,

$$\left.\begin{array}{c} (G,S) \in \Phi_d(G,S) \text{ for all } d \in D, \\ \text{and} \\ \Gamma_d(G,S) = \{G\} \text{ for all } d \notin S. \end{array}\right\} \tag{4}$$

Thus, in each state (G,S) each player d has the option of proposing that the status quo network-coalition pair be maintained and if the player is not part of the coalition whose turn it is to move, then the status quo is the only network proposal available to that player. Notice however, that even if a player is not part of the coalition whose turn it is to move, then that player can propose that another coalition, other than the status quo coalition S, be chosen to propose the next network.

Assumption A-1(i) implies that the correspondence

$$(G,S) \to \Phi(G,S) := \Pi_{d\in D}\Phi_d(G,S), \tag{5}$$

has a closed graph.

In order for players to decide which states to propose, we must specify the payoff functions. We shall assume that

A-2 (*measurability and continuity of payoffs*)

each player $d \in D$ has a payoff function

$$r_d(\cdot,\cdot) : (\mathbb{G}\times\mathcal{F})\times(\mathbb{G}\times\mathcal{F})^m \to [-M,M] \tag{6}$$

such that

(i) for each state $(G,S)\in(\mathbb{G}\times\mathcal{F})$, $r_S((G,S),\cdot)$ is continuous on $(\mathbb{G}\times\mathcal{F})^m$, and

(ii) for each m-tuple of proposals

$$(G_D,S_D)=(G_d,S_d)_{d\in D}\in(\mathbb{G}\times\mathcal{F})^m,$$

$$r_S(\cdot,(G_D,S_D)) \text{ is } B(\mathbb{G}\times\mathcal{F})\text{-measurable.}$$

Thus, if the current state is (G,S) (i.e., if the status quo network is G and it is coalition S's turn to move) and if players propose m-tuple of networks-coalition pairs

$$(G_D,S_D)\in\Phi(G,S),$$

player $\overline{d}'s$ payoff is given by

$$r_{\overline{d}}((G,S),(G_D,S_D)):=r_{\overline{d}}((G,S),(G_{\overline{d}},S_{\overline{d}},G_{-\overline{d}},S_{-\overline{d}})).$$

2.4 The Law of Motion

Given the profile of player proposals (G_D,S_D) and given the current state, $(G,S)\in(\mathbb{G}\times\mathcal{F})$, *nature* then *chooses* the next state (i.e., the next network-coalition pair) according to probabilistic transition law, $q(\cdot|(G,S),(G_D,S_D))$ defined on the state space $(\mathbb{G}\times\mathcal{F},B(\mathbb{G}\times\mathcal{F}))$. We will assume the following concerning the law of motion:

A-3 (*measurability, continuity, and domination of the law of motion*)

(i) for all $E\in B(\mathbb{G}\times\mathcal{F})$, the function

$$((G,S),(G_D,S_D))\to q(E|(G,S),(G_D,S_D))$$

is measurable over the graph of $\Phi(\cdot)$;

(ii) for all d_Ω-closed $F\in B(\mathbb{G}\times\mathcal{F})$, the function

$$(G_D,S_D)\to q(F|(G,S),(G_D,S_D))$$

is continuous on $\Phi(G,S)$ for all $(G,S)\in(\mathbb{G}\times\mathcal{F})$; and

(iii) for all $((G,S),(G_D,S_D))$ contained in the graph of $\Phi(\cdot)$ the measure

$$q(\cdot|(G,S),(G_DS_D))$$

is absolutely continuous with respect the probability measure $\mu=\nu\times\gamma$ defined on $(\Omega,B(\Omega))=(\mathbb{G}\times\mathcal{F},B(\mathbb{G}\times\mathcal{F}))$ (i.e., $q(\cdot|(G,S),(G_D,S_D))\ll\mu$ for all $((G,S),(G_D,S_D))\in Gr\Phi(\cdot)$).

Remarks 1: *In order to save writing and spare the reader, when no confusion is possible, we will use the notation*

$$(\Omega,B(\Omega))=(\mathbb{G}\times\mathcal{F},B(\mathbb{G}\times\mathcal{F}))$$

for our state space and the notation ω for elements (G,S) of the state space.

It should be noted that (A.3)(ii) is stronger than the usual weak continuity assumption. Under weak continuity, we would have for any sequence $\{(\omega_D^n)\}_n$ in $\Phi(\omega)$ with

$$(\omega_D^n) \rightarrow (\overline{\omega}_D) \in \Phi(\omega),$$

and any d_Ω-closed $F \in B(\Omega)$,

$$\limsup_n q(F|\omega, (\omega_D^n)) \leq q(F|\omega, (\overline{\omega}_D))$$

or equivalently,

$$\int_\Omega f(\omega')q(d\omega'|\omega, (\omega_D^n)) \rightarrow \int_\Omega f(\omega')q(d\omega'|\omega, (\overline{\omega}_D)),$$

for any bounded, continuous function $f(\cdot)$. Under (A.3)(ii), however, we have strengthened weak continuity so that for any sequence $\{\omega_D^n\}_n$ in $\Phi(\omega)$ with

$$\omega_D^n \rightarrow \overline{\omega}_D \in \Phi(\omega),$$

and any h-closed $F \in B(\Omega)$,

$$\lim_n q(F|\omega, \omega_D^n) = q(F|\omega, \overline{\omega}_D)$$

or equivalently (by Delbaen's Lemma (1974)),

$$\int_\Omega v(\omega')q(d\omega'|\omega, \omega_D^n) \rightarrow \int_\Omega v(\omega')q(d\omega'|\omega, \overline{\omega}_D),$$

for any bounded, *measurable* function $v(\cdot)$.

2.5 Plans and Stationary Correlated Strategies

2.5.1 Plans

A *plan* $\pi_d = (\pi_d^1, \pi_d^2, \ldots)$ for player $d \in D$ is a sequence of history dependent conditional probability measures on $(\Omega, B(\Omega))$. Under plan π_d in period n given the history of states and action $m-$*tuples* (i.e., the $(n-1)$-sequence of network-coalition pairs and m-tuples of network-coalition proposals) $H^{n-1} := \left(\omega^1, \omega_D^1, \omega^2, \omega_D^2, \ldots, \omega^{n-1}, \omega_D^{n-1}\right)$, and given the current (period n) state $\omega^n = (G^n, S^n)$, player d chooses *a* network-coalition proposal according to the conditional probability measure

$$\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{7}$$

Here, $\mathcal{P}\left(\Phi_d(\omega^n)\right)$ is the set of all probability measures with support contained in $\Phi_d(\omega^n)$.[7] Let $\mathcal{H}^{n-1}$ denote set of all $(n-1)$-histories and let

$$\Pi_d^n := \Pi_{\Phi_d}(\mathcal{H}^{n-1} \times \Omega, \mathcal{P}(\Omega))$$

denote the set of all measurable functions, $(H^{n-1}, \omega^n) \rightarrow \pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}(\Omega)$ such that $\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right)$ for all $\omega^n \in \Omega$. Formally, the set of plans for player d is given by

$$\Pi_d^\infty := \prod_{n=1}^{\infty} \Pi_d^n.$$

[7] For any set $E \subseteq \mathbb{G} \times \mathcal{F}$ we shall denote by $\mathcal{P}\left(E\right)$ the set of all probability measures with support contained in E.

A *Markov plan* $\psi_d = (\psi_d^1, \psi_d^2, \ldots)$ for player $d \in D$ is a sequence of state-dependent conditional probability measures on $(\Omega, B(\Omega))$. Under Markov plan ψ_d in period n given the current (period n) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player d chooses a network proposal according to the conditional probability measure

$$\psi_d^n(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{8}$$

Let

$$\Sigma_d^n := \Sigma_{\Phi_d}(\Omega, \mathcal{P}(\Omega)) := \Sigma_{\Phi_d}$$

denote the set of all measurable functions, $\omega \to \psi_d^n(\cdot|\omega) \in \mathcal{P}(\Omega)$ such that $\psi_d^n(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right)$ for all $\omega^n \in \Omega$. The set of Markov plans for player d is given by

$$\Sigma_d^\infty := \prod_{n=1}^{\infty} \Sigma_d^n.$$

A *stationary Markov plan* $(\sigma_d, \sigma_d, \ldots)$ for player $d \in D$ - or as we shall call it here - a stationary strategy for player $d \in D$ - is a constant sequence of state-dependent conditional probability measures on $(\Omega, B(\Omega))$. Under stationary strategy $(\sigma_S, \sigma_S, \ldots)$ given the current (period n) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player d, in each and every period n, chooses a network proposal according to the conditional probability measure

$$\sigma_d(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{9}$$

Rather than write $\sigma_d(\cdot|\omega)$ we will sometimes write $\sigma_d(\omega)$.

2.5.2 Stationary Correlated Strategies

A *stationary correlated strategy* consists of $m+1$ functions, $\lambda^i(\cdot) : \Omega \to [0,1]$ such that $\sum_{i=0}^m \lambda^i(\omega) = 1$ and $m+1$ measurable functions

$$\sigma_D^i(\cdot) : \Omega \to \underbrace{\mathcal{P}(\Omega) \times \cdots \times \mathcal{P}(\Omega)}_{|D| \text{ times}}$$

such that for each $i = 0, 1, \ldots, m$ and each state $\omega \in \Omega$, $\sigma_D^i(\omega) = (\sigma_d^i(\cdot|\omega))_{d \in D} \in \Pi_{d \in D} \mathcal{P}\left(\Phi_d(\omega)\right)$. Thus for each $i = 0, 1, \ldots, m$, $\sigma_D^i(\cdot)$ is an m-tuple of stationary strategies (recall that $|D| = m$) where each $\sigma_d^i(\cdot|\cdot)$ is an element of Σ_{Φ_d}. Under strategy m-tuple $\sigma_D^i(\cdot) = (\sigma_d^i(\cdot|\cdot), \sigma_{-d}^i(\cdot|\cdot))$, if the current state is $\omega \in \Omega$ then each player d chooses his network-coalition proposal (i.e., chooses his action) according to the probability measure $\sigma_d^i(\cdot|\omega) \in \mathcal{P}\left(\Phi_d(\omega)\right)$.

Under stationary correlated strategy $(\lambda^i(\cdot), (\sigma_d^i(\cdot|\cdot)))_{i=0}^m$, if the current state is $\omega \in \Omega$ then the i^{th} m-tuple $\sigma_D^i(\cdot)$ of stationary strategies (one for each player) is chosen with probability $\lambda^i(\omega)$ and each player $d \in D$ follows stationary strategy

$$\omega \to \sigma_d^\lambda(\cdot|\omega) := \sum_{i=0}^m \lambda^i(\omega) \sigma_d^i(\cdot|\omega) \in \mathcal{P}\left(\Phi_d(\omega)\right).$$

Thus, for each player d, $\sigma_d^\lambda(\cdot|\cdot)$ is an element of Σ_{Φ_d} and under stationary strategy $\sigma_d^\lambda(\cdot|\cdot) := \sum_{i=0}^m \lambda^i(\cdot)\sigma_d^i(\cdot|\cdot)$, if the current state is $\omega \in \Omega$ then player d will choose his network-coalition proposal according to probability measure

$$\sigma_d^\lambda(\cdot|\omega) \in \mathcal{P}\left(\Phi_d(\omega)\right).$$

Note that if $\omega = (G, S)$ and $d \notin S$, then any measure $\sigma_d(\cdot|\omega)$ in $\mathcal{P}\left(\Phi_d(\omega)\right)$ is such that

$$\sigma_d(\{G\} \times \mathcal{F}|\underbrace{G, S}_{\omega}) = 1.$$

Thus, in any state $\omega = (G, S)$, if player d is not a member of the active coalition S, then his part of the correlated proposal strategy,

$$\sigma_d^\lambda(\cdot|G, S) := \sum_{i=0}^m \lambda^i(G, S)\sigma_d^i(\cdot|G, S),$$

places probability 1 on the set of all proposals (G', S') that include the status quo network G and zero probability on all others.

2.6 Player Payoffs

Given stationary correlated strategy $(\lambda^i(\cdot), (\sigma_d^i(\cdot|\cdot)))_{i=0}^m$, if the current state is $\omega \in \Omega$ then player d's immediate expected payoff is

$$r_d(\omega, \sigma_D^\lambda(\omega)) := \int_{\Phi(\omega)} r_d(\omega, \omega_D) d\sigma_D^\lambda(\omega_D|\omega)$$

where $\sigma_D^\lambda(\omega) := \sigma_D^\lambda(\cdot|\omega)$ is the product measure $\times_d \sigma_d^\lambda(\cdot|\omega)$ with support contained in $\Phi(\omega) := \Pi_{d\in D}\Phi_d(\omega)$.

If network-coalition proposal m-tuple ω_D is chosen according to product measure $\sigma_D^\lambda(\cdot|\omega)$, then nature chooses the next network-coalition pair (i.e., the next state) according to the law of motion (i.e., the probability measure) $q(\cdot|\omega, \omega_D)$.

Let

$$r_d^n(\sigma_D^\lambda)(\omega) := r_d^n(\sigma_d^\lambda, \sigma_{-d}^\lambda)(\omega)$$

$$= \int_\Omega \left[\int_{\Phi(\omega)} r_d(\omega', \omega_D) d\sigma_D^\lambda(\omega_D|\omega')\right] q^n(\omega'|\omega, \sigma_D^\lambda(\omega))$$

denote the n^{th} period expected payoff to player d under stationary correlated strategy $\sigma_D^\lambda(\cdot)$ starting at network-coalition pair $\omega = (G, S)$ given law of motion $q(\cdot|\cdot, \cdot)$. Here, $q^n(\cdot|\omega, \sigma_D^\lambda(\omega))$ is defined recursively by

$$q^n(E|\omega, \sigma_D^\lambda(\omega))$$

$$= \int_\Omega q^{n-1}(E|\omega', \sigma_D^\lambda(\omega'))q(\omega'|\omega, \sigma_D^\lambda(\omega))$$

$$= \int_\Omega q^{n-1}(\omega'|\omega, \sigma_D^\lambda(\omega))q(E|\omega', \sigma_D^\lambda(\omega')).$$

The discounted expected payoff to player d over an infinite time horizon under stationary correlated strategy $\sigma_D^\lambda(\cdot) \in \prod_{d\in D} \Sigma_{\Phi_d}$ starting at state ω is then given by

$$E_d(\sigma_D^\lambda)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(\sigma_D^\lambda)(\omega).$$

In general, the discounted expected payoff to player d over an infinite time horizon under plan $\pi_D = (\pi_d)_{d\in D} \in \Pi^\infty := \prod_{d\in D} \Pi_d^\infty$ starting in state ω is then given by

$$E_d(\pi_D)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(\pi_D)(\omega).$$

3 Dynamic Network and Coalition Formation Games and Nash Equilibrium

A dynamic network and coalition formation game is given by

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D}.$$

A dynamic network and coalition formation game starting at state $\omega \in \Omega$ is given by

$$\Gamma_\omega := (\Omega, E_d(\cdot)(\omega), \Pi_d^\infty)_{d\in D}.$$

Definition 4 (*Nash Equilibrium*)

A stationary correlated strategy $(\lambda^{*i}(\cdot), (\sigma_d^{*i}(\cdot|\cdot))_{d\in D})_{i=0}^m$ with corresponding m-tuple of stationary strategies $\sigma_D^{*\lambda}(\cdot) = (\sigma_d^{*\lambda}(\cdot|\cdot))_{d\in D}$ is a Nash equilibrium of the dynamic network and coalition formation game Γ if for all starting network-coalition pairs $\omega = (G, S) \in \mathbb{G}\times\mathcal{F}$ and all players $d \in D$,

$$E_d(\sigma_d^{*\lambda}, \sigma_{-d}^{*\lambda})(\omega) \geq E_d(\pi_d, \sigma_{-d}^{*\lambda})(\omega) \text{ for all } \pi_d \in \Pi_d^\infty.$$

Thus, a stationary correlated strategy $(\lambda^{*i}(\cdot), (\sigma_d^{*i}(\cdot|\cdot))_{d\in D})_{i=0}^m$ with corresponding m-tuple of stationary strategies $\sigma_D^{*\lambda}(\cdot)$ is a Nash equilibrium of dynamic network and coalition formation game Γ if it is a Nash equilibrium for the game Γ_ω for all starting states.

Theorem 1 (*The Existence of Nash Equilibrium in Stationary Correlated Network and Coalition Formation Strategies*)

Under assumptions [A-1]-[A-3] the dynamic network and coalition formation game

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D}.$$

has a Nash equilibrium in stationary correlated strategies.

Our approach to proving existence essentially follows the approach introduced by Nowak and Raghavan in their seminal 1992 paper. But because we assume that players' discount factors β_d are heterogenous, and more importantly, because our stochastic continuity assumptions concerning the law of motion are slightly weaker than those of Nowak and Raghavan (their assumptions imply our assumptions), for the convenience of the reader we include a proof in the last section of the paper. The basic objective of the proof is to show that there exists a stationary correlated strategy $(\lambda^{*i}(\cdot), (\sigma_d^{*i}(\cdot|\cdot))_{d \in D})_{i=0}^m$ with corresponding m-tuple of stationary strategies $(\sigma_d^{*\lambda}(\cdot|\cdot))_{d \in D}$ and an m-tuple of $B(\Omega)$-measurable value functions, $w_d^*(\cdot) : \Omega \rightarrow [-M, M]$, such that for each player $d \in D$ and for all states $\omega \in \Omega$,

$$w_d^*(\omega) = r_d(\omega, \sigma_D^{*\lambda}(\omega)) + \beta_d \int_\Omega w_S^*(\omega')q(\omega'|\omega, \sigma_D^{*\lambda}(\omega)),$$

where

$$r_d(\omega, \sigma_D^{*\lambda}(\omega)) = \int_{\Phi(\omega)} r_S(\omega, \omega'_D) d\sigma^{*\lambda}(\omega'_D|\omega),$$

and

$$\int_\Omega w_d^*(\omega')q(\omega'|\omega, \sigma_D^{*\lambda}(\omega)) = \int_\Omega \int_{\Phi(\omega)} w_d^*(\omega')dq(\omega'|\omega, \omega'_D)d\sigma_D^{*\lambda}(\omega'_D|\omega).$$

4 The Equilibrium Markov Process: Definitions, Terminology, and Basic Properties

4.1 The Equilibrium Process

Under stationary correlated equilibrium, $\sigma_D^{*\lambda}(\cdot) = (\sigma_d^{*\lambda}(\cdot|\cdot))_{d \in D}$, the emergent Markov process of network and coalition formation,

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^\infty,$$

is governed by the equilibrium Markov transition,[8]

$$p^*(E|\omega) = \textstyle\int_E dq(\omega'|\omega, \sigma_D^{*\lambda}(\omega))$$

$$= \textstyle\int_{\Phi(\omega)} \int_E dq(\omega'|\omega, \omega'_D) d\sigma^{*\lambda}(\omega'_D|\omega)$$

$$= \textstyle\int_{\Phi(\omega)} q(E|\omega, \omega'_D) d\sigma^{*\lambda}(\omega'_D|\omega).$$

Thus,

$$\Pr\left\{W_{n+1}^* | W_n^* = \omega\right\} = p^*(E|\omega)$$

and

$$\Pr\left\{W_n^* \in E | W_0^* = \omega\right\} = p^{*n}(E|\omega) = q^n(E|\omega, \sigma_D^{*\lambda}(\omega)),$$

[8] Law of motion $\omega \rightarrow p^*(\cdot|\omega)$ is a Markov transition if for each ω, $p^*(\cdot|\omega)$ is a probability measure and for each $E \in B(\Omega)$,

$$p^*(E|\cdot) : \Omega \rightarrow [0, 1]$$

is measurable. Here, $\omega = (G, S)$ is a realization of the process $W_n^* = (G_n^*, S_n^*)$ for some n.

where the n-step transition $p^{*n}(\cdot|\cdot)$ is defined recursively as follows: for all $\omega \in \Omega$ and $E \in B(\Omega)$,

$$p^{*n}(E|\omega) = \int_{\Omega} p^*(E|\omega')p^{*n-1}(d\omega'|\omega) = \int_{\Omega} p^{*n-1}(E|\omega')p^*(d\omega'|\omega) \tag{10}$$

for $n = 1, 2, \ldots$, and $p^{*0}(\cdot|\omega) = \delta_{\omega}(\cdot)$ is the Dirac measure at ω.

4.2 Absorbing Sets and Invariant and Ergodic Probability Measures

A probability measure $\lambda(\cdot)$ on the state space of feasible network-coalition pairs $(\Omega, B(\Omega))$ is invariant for Markov transition $p^*(\cdot|\cdot)$ (i.e., is p^*-invariant) if

$$\lambda(E) = \int_{\Omega} p^*(E|\omega)d\lambda(\omega) \text{ for any } E \in B(\Omega). \tag{11}$$

Thus, if probability measure $\lambda(\cdot)$ is p^*-invariant, then for any set of network-coalition pairs $E \in B(\Omega)$, if the current status quo network-coalition pair $\omega_n = (G_n, S_n)$ is chosen according to probability measure $\lambda(\cdot)$ - so that the probability that ω_n lies in E is just $\lambda(E)$ - then the probability that next period's network-coalition pair $\omega_{n+1} = (G_{n+1}, S_{n+1})$ lies in E is also $\lambda(E) = \int_{\Omega} p^*(E|\omega)d\lambda(\omega)$. Denote by $\mathcal{I}^*$ the collection of all p^*-invariant measure.

Let $\mathcal{L}^* \subseteq B(\Omega)$ denote the collection of all p^*-absorbing sets (i.e., $E \in \mathcal{L}^*$ if and only if $p^*(E|\omega) = 1$ for all network-coalition pairs $\omega \in E$). Note that the set of all absorbing sets is closed under countable unions and intersections,

A p^*-absorbing set $E \in \mathcal{L}^*$ is said to be *indecomposable* if it does not contain the union of two disjoint absorbing sets. We say that an absorbing set $A \in \mathcal{L}^*$ is *atomic* if there does not exist another absorbing set $E \in \mathcal{L}^*$ such that E is a proper subset of A, i.e., such that $A \backslash E \neq \emptyset$. We will denote by $\mathcal{A}^*$ the set of all atomic absorbing sets. Thus, the set of atomic absorbing sets is given by

$$\mathcal{A}^* = \{A \in \mathcal{L}^* : \text{ there does not exist } E \in \mathcal{L}^* \text{ with } E \subset A \text{ and } A \backslash E \neq \emptyset \}. \tag{12}$$

Note that if $A \in \mathcal{A}^*$ and $A' \in \mathcal{A}^*$, then $A \cap A' = \emptyset$, and if $E \in \mathcal{L}^*$ and $A \in \mathcal{A}^*$, then either $E \cap A = \emptyset$ or $E \cap A = A$. We will be most interested in the subset of atoms each of whose states (network-coalition pairs) is expected to have an infinite number of visitations by the network-coalition formation process starting from any network-coalition pair. The number of visitations to atom $A \in \mathcal{A}^*$ by process $\{W_n^*\}_n$ is given by

$$\eta_A = \sum_{n=1}^{\infty} I_A(W_n^*). \tag{13}$$

The expected number of visitations starting from network-coalition pair $\omega = (G, S)$ is given by

$$E_{\omega}[\eta_A] = \sum_{n=1}^{\infty} p^{*n}(A|\omega). \tag{14}$$

We say that the set A is recurrent if $E_\omega[\eta_A] = \infty$, meaning that the set has an infinite number of visitations. Atoms each of whose members has an infinite number of visitations will be denoted by $\mathcal{A}^{\infty *}$. We will call these atoms *infinite atoms.* Thus, the set of all infinite atoms is given by

$$\mathcal{A}^{\infty *} = \left\{ A \in \mathcal{A}^* : E_\omega[\eta_{\{\omega\}}] = \sum_{n=1}^{\infty} p^{*n}(\{\omega\} | \omega) = \infty \text{ for all } \omega \in A \right\}. \tag{15}$$

A p^*-invariant measure $\lambda(\cdot)$ is said to be p^*-ergodic if $\lambda(E) = 0$ or $\lambda(E) = 1$ for all $E \in \mathcal{L}^*$. Denote by $\mathcal{E}^*$ the collection of all p^*-ergodic measures.

Because the p^*-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{I}^*$ of p^*-invariant measures (see Theorem 19.25 in Aliprantis and Border (1999)), each measure $\lambda(\cdot)$ in $\mathcal{I}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.

4.3 Hitting Probabilities, Irreducibility, and Maximal Harris Sets

Often we will be interested in determining the probability with which the network-coalition formation process reaches a particular set of network-coalition pairs. In particular, let

$$\tau_E^* := \inf\{n \geq 1 : (G_n^*, S_n^*) \in E\} = \inf\{n \geq 1 : W_n^* \in E\}$$

be the *hitting time* of network-coalition formation process $\{W_n^*\}_n$ for set $E \in B(\Omega)$, and following in Tweedie (2001), let

$$L^*(\omega, E) := \Pr\{\tau_E^* < \infty | W_0^* = \omega\} = \Pr\{\cup_{n=1}^{\infty} (W_n^* \in E | W_0^* = \omega)\} \tag{16}$$

denote the probability of hitting (or reaching) in finite time the set of network-coalition pairs E starting from network-coalition pair $\omega \in \Omega$ given transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$.

Also, we will often be interested in determining the probability with which the network-coalition formation process reaches a particular set of network-coalition pairs infinitely often (denoted by i.o.). This probability is given by

$$\left.\begin{aligned} Q^*(E|\omega) &:= \Pr\{W_n^* \in E \text{ i.o.} | W_0^* = \omega\} \\ &= \Pr\{\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} (W_n^* \in E | W_0^* = \omega)\} \text{ for all } \omega \in \Omega. \end{aligned}\right\} \tag{17}$$

By the Orey (1971), we know that

$$\text{if for any } E \in B(\Omega),\ L^*(\omega, E) = 1 \text{ for all } \omega \in \Omega, \text{ then } Q^*(E|\omega) = 1 \text{ for all } \omega\Omega. \tag{18}$$

The network-coalition formation process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is said to be ψ-irreducible if for some nontrivial measure $\psi(\cdot)$ on $B(\Omega)$,

$$\psi(E) > 0 \text{ implies } L^*(\omega, E) > 0 \text{ for all } \omega \in \Omega.$$

Thus if the process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is ψ-irreducible, then it hits all the "important" sets of network-coalition pairs (i.e., the sets E such that $\psi(E) > 0$) with positive probability starting from any network-coalition pair in the state space $\Omega = \mathbb{G} \times \mathcal{F}$.

A set of network-coalition pairs $H \in B(\Omega)$ is called a *Maximal Harris set* if

(i) there exists a measure $\varphi(\cdot)$ on $B(\Omega)$ with $\varphi(H) > 0$ such that $\varphi(A) > 0$ implies $L^*(\omega, A) = 1$ for all $\omega \in H$, and

(ii) every network-coalition $\omega = (G, S)$ such that $L^*(\omega, H) = 1$ is contained in H.

Note that Maximal Harris sets are absorbing; that is, $p^*(H|\omega) = q(H|\omega, \sigma_D^{*\lambda}(\omega)) = 1$ for all network-coalition pairs $G \in H$. Moreover, if H and H' are distinct Maximal Harris sets, then they are disjoint.

A set of network-coalition pairs $T \in B(\Omega)$ is *transient* if T is the disjoint union of countably many *uniformly transient sets* U_j, that is, sets $U_j \in B(\Omega)$ such that $T = \cup_j U_j$ and for each set there is a finite constant M_j, such that for all network-coalition pairs $\omega \in U_j$,

$$E_\omega[\eta_{U_j}] = \sum_{n=1}^{\infty} p^{*n}(U_j|\omega) < M_j. \tag{19}$$

A set of network-coalition pairs $E \in B(\Omega)$ is said to be p^*-*inessential* if

$$Q^*(E|\omega) = 0 \text{ for all } \omega \in \Omega. \tag{20}$$

Thus, a set of states E is inessential if the probability that the network-coalition formation process visits the set E infinitely often is zero stating from any state. If a set of states is inessential, then if the process visits the state at all, it leaves the state for good after finitely many moves. Let

$$\mathcal{M}^* = \{E \in B(\Omega) : Q^*(E|\omega) = 0 \text{ for all } \omega \in \Omega\}, \tag{21}$$

denote the inessential states. The union of countable many inessential states is called an *improperly* p^*-*essential set*. Any other set is called properly p^*-*essential*.

4.4 The Fundamental Conditions for Stability: Drift and Global Uniform Countable Additivity

Given the Markov transition $\omega \to p^*(\cdot|\omega)$ what can be said concerning stability? Quite a bit if the Markov transition $p^*(\cdot|\cdot)$ satisfies the following two conditions:

The Tweedie Conditions (2001):

there exists a measurable set of network-coalition pairs $C \subseteq \Omega$, a nonnegative measurable function

$$V(\cdot) : \Omega \to [0, \infty],$$

and a finite b such that (i) (the drift condition) for all $\omega \in \Omega$

$$\int_{\Omega} V(\omega')dp^*(\omega'|\omega) \leq V(\omega) - 1 + bI_C(\omega),$$

and (ii) (uniform countable additivity) for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n\to\infty} \sup_{\omega \in C} p^*(B_n|\omega) = 0.$$

We shall say that the Markov transition $p^*(\cdot|\cdot)$ satisfies *global uniform countable additivity* if for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n\to\infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0, \tag{22}$$

and we will say that the Tweedie conditions are satisfied globally if Tweedie conditions hold with $C = \Omega$.

We will show in Section 5 below, using some beautiful results by Meyn and Tweedie (1993a), Chen and Tweedie (1997), Tweedie (2001) and Costa and Dufour (2005), that if the emergent Markov transition $p^*(\cdot|\cdot)$ governing the equilibrium process of network and coalition formation is *globally uniformly countable additive*, then the equilibrium process possesses some striking stability properties - similar to those demonstrated in Page and Wooders (2007) for static abstract games of network formation.

To begin let us strengthen slightly our stochastic continuity assumption A-3(ii) as follows:

A-3 (ii)' for all d_Ω-closed $F \in B(\Omega)$, the function

$$(\omega, \omega_D) \to q(F|\omega, \omega_D)$$

is continuous over the graph of $\Phi(\cdot)$.

Theorem 2 *(Setwise Convergence on Closed Sets and Global Uniform Countable Additivity)*

Given that the state space $(\Omega, B(\Omega))$ of networks and coalitions is a compact metric space, if the law of motion is such that $q(F|\cdot,\cdot)$ is continuous on the graph of $\Phi(\cdot)$ for all d_{Ω}-closed sets F of network-coalition pairs (i.e., if A-3(ii)' is satisfied), then $p^*(\cdot|\cdot)$ is globally uniformly countable additive.

Proof. Let $M(\Omega)$ denote the Banach space of bounded measurable functions on $(\Omega, B(\Omega))$, equipped with the sup norm and let $rca(\Omega)$ denote the Banach space of finite signed Borel measures on $(\Omega, B(\Omega))$. First, observe that the set of probability measures

$$\Pi_{\Phi} := \{q(\cdot|\omega, \omega_D) : (\omega, \omega_D) \in Gr\Phi(\cdot)\}$$

is sequentially compact in the $\sigma(rca(\Omega), M(\Omega))$ topology. This follows because $Gr\Phi(\cdot)$ is a compact metric space and because by Delbaen's Lemma (1974),

$$(\omega^n, \omega_D^n) \to (\overline{\omega}, \overline{\omega}_D))$$

implies that

$$\int_{\Omega} v(\omega')q(d\omega'|\omega^n, \omega_D^n) \to \int_{\Omega} v(\omega')q(d\omega'|\overline{\omega}, \overline{\omega}_D)) \text{ for all } v(\cdot) \ \in M(\Omega).$$

By Corollary 2.2 in Lasserre (1998), therefore,

$$\lim_{k\to\infty} \sup_{(\omega,\omega_D)\in Gr\Phi(\cdot)} \int_{\Omega} v_k(\omega')q(d\omega'|\omega, \omega_D) = 0 \tag{23}$$

whenever $v_k(\cdot) \downarrow 0$, $v_k(\cdot) \in M(\Omega)$.

To see that (23) implies global uniform countable additivity (22), consider a sequence $\{B_k\}_k \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_k \downarrow \emptyset$) and let $v_k(\cdot) := I_{B_k}(\cdot)$, where

$$I_{B_k}(\omega) = \begin{cases} 1 & \text{if } \omega \in B_k \\ 0 & \text{if } \omega \notin B_k. \end{cases}$$

We have $I_{B_k}(\cdot) \downarrow 0$, $I_{B_k}(\cdot) \in M(\Omega)$ and

$$\int_{\Omega} v_k(\omega')q(d\omega'|\omega, \omega_D) = q(B_k|\omega, \omega_D).$$

Finally, for each k let $(\omega^k, \omega_D^k) \in Gr\Phi(\cdot)$ be such that

$$q(B_k|\omega^k, \omega_D^k) = \sup_{(\omega,\omega_D)\in Gr\Phi(\cdot)} q(B_k|\omega, \omega_D).$$

We have for all $\omega \in \mathbb{G}$,

$$p^*(B_k|\omega) = \int_{\Phi(\omega)} q(B_k|\omega, \omega'_D)d\sigma^{*\lambda}(\omega'_D|\omega) \leq q(B_k|\omega^k, \omega_D^k) \to 0.$$

■

Remarks 2: *Alternatively, global uniform countable additivity will be guaranteed if instead of assuming A-3(ii)', we add to our list of assumptions A-3 the following assumption:*

A-3 (iv) the densities $f(\cdot|\omega,\omega_D)$ of $q(\cdot|\omega,\omega_D)$ with respect to the dominating probability measure μ are integrably bounded, that is, there exists a μ-integrable function

$$h(\cdot):\Omega\rightarrow R_+$$

such that for all $(\omega,\omega_D)\in Gr\Phi(\cdot)$,

$$0\leq f(\omega'|\omega,\omega_D)\leq h(\omega')\text{ for all }\omega'\in\Omega. \tag{24}$$

With this additional assumption, we have for any sequence $\{B_n\}_n\subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n\downarrow\emptyset$),

$$p^*(B_n|\omega)=\int_{B_n}dq(\omega'|\omega,\sigma_D^{*\lambda}(\omega))$$

$$=\int_{\Phi(\omega)}\int_{B_n}dq(\omega'|\omega,\omega'_D)d\sigma_D^{*\lambda}(\omega'_D|\omega))$$

$$=\int_{\Phi(G)}\left(\int_{B_n}f(\omega'|\omega,\omega'_D)d\mu(\omega')\right)d\sigma_D^{*\lambda}(\omega'_D)|\omega))$$

$$\leq\int_{B_n}h(\omega')d\mu(\omega')\rightarrow 0\text{ as }B_n\downarrow\emptyset.$$

Let A-3$'$ denote the altered or augmented set of assumptions A-3 (i.e., either altered by A-3 (ii)' or augmented by A-3 (iv)).

By Theorem 2, under assumptions A-1, A-2, A-3', the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of network and coalition formation satisfies global uniform countable additivity. As a consequence by letting C equal the entire state space Ω, $V(\omega)=1$ for all $\omega\in\Omega$, and $b=2$, it is easy to see that the drift condition in the Tweedie conditions is also satisfied. Thus by Theorem 2, the Tweedie conditions are satisfied globally (i.e., with $C=\Omega$), and thus by strengthening slightly the stochastic continuity conditions the law of motion $q(\cdot|\cdot,\cdot)$ must satisfy in the first place to guarantee the existence of an equilibrium Markov transition, $p^*(\cdot|\cdot)$, we will be able to show that even though uncountably many networks may form, the equilibrium process of network and coalition formation possesses only a finite number of basins of attraction.

5 Basins of Attraction, Invariance, and Ergodicity

We now have our first result concerning stochastic basins of attraction and the stability of the emergent network-coalition formation process

$$\{W_n^*\}_n=\{(G_n^*,S_n^*)\}_{n=1}^{\infty}$$

governed by $p^*(\cdot|\cdot)$.

Theorem 3 (*Basins of Attraction: The Atomic Decomposition of the State Space*)

Under assumptions [A-1], [A-2] and [A-3'], the emergent network-coalition formation process

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$ *generates a unique finite set of infinite atoms* $\mathcal{A}^{\infty *}$ *and a unique finite decomposition of the state space of network-coalition pairs* $\Omega = \mathbb{G} \times \mathcal{F}$ *given by*

$$\Omega = \left(\cup_{i=1}^{N} A_i\right) \cup T, \tag{25}$$

where $\{A_1, \ldots, A_N\} = \mathcal{A}^{\infty *}$ *is the set of infinite atoms and* T *is improperly essential. Moreover,* $\mathcal{A}^{\infty *} = \mathcal{A}^*$ *(all atoms are infinite), each atom is a Maximal Harris set, and*

$$L^*(\omega, \cup_i A_i) = 1 \tag{26}$$

for every network-coalition pair $\omega \in \Omega$.

Proof. Because the Tweedie conditions hold globally under our assumptions [A-1], [A-2] and [A-3'], by Theorem 2 in Tweedie (2001), the state space Ω admits a finite decomposition

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T_H$$

where each H_i is indecomposable and Maximal Harris and T_H is transient. Moreover this Harris decomposition is such that $L^*(\omega, \cup_{i=1}^{N} H_i) = 1$ for all $\omega \in \Omega$. By Jamison (1972) and Winkler (1975) in fact each Harris set H_i is properly essential and T_H is also improperly essential. Because the state space of network-coalition pairs is a compact metric space, and therefore second countable, it follows from Theorem 5.8 in Chen and Tweedie (1997) (see the proof) that each Harris set is given uniquely as

$$H_i = A_i \cup E_i$$

where A_i is an atom and E_i is improperly essential. Because H_i is indecomposable and A_i is an atom, E_i is indecomposable. Therefore by Theorem 6 in Jain and Jamison (1967), E_i is transient, and it follows from Theorem 2 in Tweedie (2001) that because H_i is p^*-absorbing, E_i being transient implies that $L^*(\omega, A_i) = 1$ for all $\omega \in H_i$. Moreover, by Proposition 5.3 in Chen and Tweedie (1997), each atom A_i is an infinite atom. Thus we have the desired unique, atomic decomposition

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T_H = \left(\cup_{i=1}^{N} (A_i \cup E_i)\right) \cup T_H = \left(\cup_{i=1}^{N} A_i\right) \cup \left(\cup_{i=1}^{N} E_i\right) \cup T_H, \quad (*)$$

where the transient set T is given by $\left(\cup_{i=1}^{N} E_i\right) \cup T_H$. Moreover, because $L(\omega, A_i) = 1$ for all $\omega \in H_i$ and $L(\omega, \cup_{i=1}^{N} H_i) = 1$ for all $\omega \in \Omega$, we have $L^*(\omega, \cup_i A_i) = 1$ for all $\omega \in \Omega$.

Because T is transient and the A_i are atoms, we conclude that all the atoms in $\mathcal{A}^*$ must be used in the decomposition (*) and since each atom in the decomposition must be infinite, we must also conclude that $\mathcal{A}^{\infty *} = \mathcal{A}^*$. Finally, by Theorem 3 of in Jain and Jamison (1967), because each atom A_i is indecomposable and essential, each A_i is maximal Harris. ■

Each infinite atom forms a basin of attraction of the emergent Markov network and coalition formation process governed by the equilibrium transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$. In particular, starting at any network-coalition pair not contained in an absorbing atom, the network and coalition formation process will reach some atom in a finite number of moves with probability 1. Moreover, once the process has entered a particular atom it stays there with probability 1. *The remarkable fact here is that even though there are possibly uncountably many networks, there is only* ***finitely*** *many atoms - and they are unique to the equilibrium process determined by the dynamic, stochastic game of network and coalition formation.* A analogous conclusion is reached in Page and Wooders (2008) for static, abstract games of network formation over finitely many networks. There it is shown that no matter what rules of network formation prevail, given any profile of player preferences the feasible set of networks contains a finite, disjoint collection of sets each set representing a *strategic* basin of attraction in the sense that if the game is repeated - each time starting at the status quo network reached in the previous play of the game - the process of network formation generated by repeating this static game will reach a network contained in some strategic basin and once there will stay there. Thus, these strategic basin of attraction represent the absorbing atoms of the static game of network formation.

The Harris decomposition in Theorem 3 is also a Doeblin decomposition (see Meyn and Tweedie (1993b) and Tweedie (1976)).

Theorem 4 (*Invariance and Ergodicity of the Process of Network and Coalition Formation)*

Suppose assumptions [A-1], [A-2] and [A-3'] hold. Let

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$, *and let*

$$\Omega = \left(\cup_{i=1}^{N} A_i\right) \cup T,$$

be the corresponding unique atomic decomposition where $\{A_1, \ldots, A_N\} = \mathcal{A}^{\infty *}$ *is the set of infinite atoms and* T *is improperly essential.*

The following statements are true:

(1) Corresponding to each basin of attraction A_i*, there is a unique* p^**-invariant probability measure* $\lambda_i(\cdot)$ *with* $\lambda_i(A_i) = 1$*. Moreover, For each network-coalition pair* $\omega = (G, S)$*,*

$$p^{*(n)}(E|\omega) := \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \overset{n}{\to} \sum_{i=1}^{N} L^*(\omega, A_i)\lambda_i(E \cap A_i), \text{ for all } E \in B(\Omega). \tag{27}$$

where $p^{*k}(E|\omega)$ *is defined recursively, see (10).*

(2) The set of all ergodic probability measures is given by

$$\mathcal{E}^* = \{\lambda_i(\cdot)\}_{i=1}^N .$$

Moreover, a probability measure $\lambda(\cdot)$ *on* $(\Omega, B(\Omega))$ is p^**-invariant, i.e.* $\lambda(\cdot) \in \mathcal{I}^*$*, if and only if*

$\lambda(\cdot)$ *is given by*

$$\lambda(E) = \sum_i^N \lambda(A_i)\lambda_i(E \cap A_i), \text{ for all } E \in B(\Omega). \tag{28}$$

(3) $\mathcal{E}^*$ *is a singleton (i.e.,* $\mathcal{E}^* = \{\lambda(\cdot)\}$*) if and only if the network-coalition formation process* $\{W_n^*\}_n$ *is* ψ*-irreducible, in which case for each network-coalition pair* $\omega = (G, S)$ *and for every set of network-coalition pairs* $E \in B(\Omega)$

$$\frac{1}{n}\sum_{k=1}^n p^{*k}(E|\omega) \xrightarrow{n} \lambda(E).$$

Proof. (1) Under our assumptions [A-1], [A-2] and [A-3'] (see the proof of Theorem 2 above), $p^*(\cdot|\cdot)$ satisfies the Tweedie conditions globally. As a result, the first statement in part (1) is an immediate consequence of Lemma 5 in Tweedie (2001). The second statement also follows from the Tweedie conditions holding globally and Theorem 1 in Tweedie (2001) (also, see Chapter 13 in Meyn and Tweedie (1993a)).

(2) Again because the Tweedie Conditions are satisfied globally, the first statement in part (2) follows from Lemma 2 in Tweedie (2001), Theorem 2.18 in Costa and Dufour (2005), part (1) of this Theorem, and Theorem 3.8 in Costa and Dufour, and the proof of Proposition 5.3 in Costa and Dufour. For the second statement in part (2): $\lambda(\cdot) \in \mathcal{I}^*$ implies (28) follows from the proof of Proposition 5.3 in Costa and Dufour. The fact that (28) implies $\lambda(\cdot) \in \mathcal{I}^*$ follows from observation (but also, see Theorem 19.25 in Aliprantis and Border (1999)).

(3) Finally, because the Tweedie Conditions are satisfied globally, necessary and sufficient conditions for $\mathcal{E}^*$ to be a singleton, given in terms of ψ-irreducibility follow from Theorem 3 in Tweedie (2001). The convergence result in part (3) follows from the convergence result in part (1) of the Theorem and the fact that if there is only one basin of attraction A (i.e., one atom or equivalently, one maximal Harris set), then by Theorem 3, $L^*(\omega, H) = 1$ for all $\omega \in \Omega$. ■

Note that the probability measures in $\mathcal{E}^*$ are *orthogonal*, that is, for all i and i' in $\{1, 2, \ldots, N\}$ with $i \neq i'$,

$$\lambda_i(\Omega \backslash A_i) = \lambda_{i'}(A_i) = 0.$$

5.1 Ergodic Properties of the Strategic Values

For each starting network-coalition pair $\omega = (G, S) \in \Omega$, $w_d^*(\omega)$ is the strategic value to player d of following his part of the stationary correlated equilibrium strategies

$\sigma_D^{*\lambda}(\cdot)$, given that all other players follow their parts of the strategy. Because $\sigma_D^{*\lambda}(\cdot)$ is Nash, we know this is the best that player d can do relative to all other strategies, even those that are history dependent. Strategies $\sigma_D^{*\lambda}(\cdot)$ together with the trembles of nature determine the equilibrium Markov process of network and coalition formation via the transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$. The questions we wish to address in this section concern the properties of players' strategic values across time and states given the equilibrium process of network and coalition formation.

We begin by considering time averages. Let

$$p^{*(n)}w_d^*(\omega) = \frac{1}{n}\sum_{k=1}^{n}\int_{\Omega} w_d^*(\omega')p^{*k}(d\omega'|\omega) = \int_{\Omega} w_d^*(\omega')p^{*(n)}(d\omega'|\omega),$$

where recall,

$$w_d^*(\omega) = E_d(\sigma_D^{*\lambda})(\omega) := \textstyle\sum_{n=1}^{\infty}\beta_d^{n-1}r_d^n(\sigma_D^{*\lambda})(\omega)$$

$$= r_d(\omega, \sigma_D^{*\lambda}(\omega)) + \beta_d \textstyle\int_{\Omega} w_d^*(\omega')dq(\omega'|\omega, \sigma_D^{*\lambda}(\omega))$$

and

$$p^{*(n)}(E|\omega) = \tfrac{1}{n}\textstyle\sum_{k=1}^{n} p^{*k}(E|\omega) = \tfrac{1}{n}\sum_{k=1}^{n}\int_{\Omega} p^*(E|\omega')p^{*k-1}(d\omega'|\omega).$$

Here, $p^{*k}(E|\omega)$ is the probability that process reaches the set of network-coalition pairs E starting at network-coalition pair $\omega = (G, S)$ in k periods or moves.

The function $p^{*(n)}w_d^*(\cdot)$ specifies for each starting network-coalition pair, player d's n-period time average expected strategic value (i.e., the average value of following his part of the stationary correlated equilibrium strategies $\sigma_D^{*\lambda}(\cdot)$ for n moves). We can think of $\lim_n p^{*(n)}w_d^*(\cdot)$ therefore as specifying for each starting network-coalition pair, player d's time average expected value.

By part (4) of Theorem 4 above, we have *for all* $\omega \in \Omega$ and $E \in B(\Omega)$

$$p^{*(n)}(E|\omega) = \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L^*(\omega, A_i)\lambda_i(E\cap A_i) = \lambda_\omega(E), \tag{29}$$

where $\lambda_\omega(\cdot) \in \mathcal{I}^*$ for all $\omega \in \Omega$ and $\{\lambda_i(\cdot) : i = 1, 2, \ldots, N\} = \mathcal{E}^*$. Because $p^{*(n)}(\cdot|\omega)$ converges setwise for all ω, by Delbaen's Lemma (1974) we have *for all* $\omega \in \Omega$

$$p^{*(n)}w_d^*(\omega) \to \sum_{i=1}^{N} L^*(\omega, A_i)\int_{H_i} w_d^*(\omega')d\lambda_i(\omega'). \tag{30}$$

Thus, we obtain one of the fundamental principles of equilibrium dynamics: the equality of time averages and state averages.

Theorem 5 *(The Equality of Time Average Values and State Average Values)*

Under assumptions [A-1], [A-2] and [A-3'] the emergent network-coalition formation process

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$ *is such that:*

(1) for each player d starting at any network-coalition pair $\omega = (G, S)$ contained in a basin of attraction A_i the time average value of the equilibrium strategies $\sigma_D^{*\lambda}$ is equal to state average value of the equilibrium strategies, that is, for all basins of attraction A_i and for all initial states $\omega = (G, S) \in A_i$,

$$\underbrace{\lim_n p^{*(n)} w_d^*(\omega)}_{time\ average} = \underbrace{\int_{A_i} w_d^*(\omega') d\lambda_i(\omega')}_{state\ average}. \tag{31}$$

Moreover, for all initial states $\omega = (G, S) \in \Omega$,

$$\lim_n p^{*(n)} w_d^*(\omega) = \sum_{i=1}^{N} L^*(\omega, A_i) \int_{A_i} w_d^*(\omega') d\lambda_i(\omega') \tag{32}$$

(2) For all invariant measures $\lambda(\cdot) \in \mathcal{I}^*$

$$\int_\Omega f_d^*(\omega') d\lambda(\omega') = \int_\Omega w_d^*(\omega') d\lambda(\omega'), \tag{33}$$

where

$$f_d^*(\omega) := \sum_{i=1}^{N} L^*(\omega, A_i) \int_{A_i} w_d^*(\omega') d\lambda_i(\omega') \text{ for all } \omega \in \Omega. \tag{34}$$

Proof. (1) Part (1) is an immediate consequence of part (4) of Theorem 4, Delbaen's Lemma (1974), and the fact that for all basins A_i and all states $\omega \in A_i$, $L^*(\omega, A_i) = 1$.

(2) Let invariant probability measure $\lambda(\cdot) = \sum_{i=1}^N \lambda(A_i)\lambda_i(\cdot) \in \mathcal{I}^*$ be given. We have

$$\int_\Omega w_d^*(\omega') d\lambda(\omega') = \sum_{i=1}^N \lambda(A_i) \int_\Omega w_d^*(\omega') d\lambda_i(\omega') = \sum_{i=1}^N \lambda(A_i) \int_{A_i} w_d^*(\omega') d\lambda_i(\omega')$$

and

$$\int_\Omega f_d^*(\omega') d\lambda(\omega') = \sum_{i=1}^N \lambda(A_i) \int_\Omega f_d^*(\omega') d\lambda_i(\omega') = \sum_{i=1}^N \lambda(A_i) \int_{A_i} f_d^*(\omega') d\lambda_i(\omega')$$

Letting $\int_{A_i} w_d^*(\omega') d\lambda_i(\omega') := w_d^*(A_i)$, we have

$$\int_{A_i} f_d^*(\omega') d\lambda_i(\omega') = \int_{A_i} \left[\sum_{i=1}^N L^*(\omega', A_i) w_d^*(A_i)\right] d\lambda_i(\omega').$$

Moreover, because for all $\omega' \in A_i$, $L^*(\omega', A_i) = 1$ and $L^*(\omega', A_{i'}) = 0$, for all $i' \neq i$,

$$\int_{A_i} \left[\sum_{i=1}^N L^*(\omega', A_i) w_d^*(H_i)\right] d\lambda_i(\omega') = w_d^*(A_i) = \int_{A_i} w_d^*(\omega') d\lambda_i(\omega').$$

Thus we have for each i

$$\int_{A_i} f_d^*(\omega')d\lambda_i(\omega') = \int_{A_i} w_d^*(\omega')d\lambda_i(\omega'),$$

and thus,

$$\int_\Omega f_d^*(\omega')d\lambda(\omega') = \sum_{i=1}^N \lambda(A_i) \int_{A_i} f_d^*(\omega')d\lambda_i(\omega')$$

$$= \sum_{i=1}^N \lambda(A_i) \int_{A_i} w_d^*(\omega')d\lambda_i(\omega')$$

$$= \int_\Omega w_d^*(\omega')d\lambda(\omega').$$

■

Also see Birkhoff's Ergodic Theorems (pointwise and mean), for example, Theorems 2.3.4 and 2.3.5 in Hernandez-Lerma and Lasserre (2003)).

By part (1) of Theorem 4, each player's time average value $\lim_n p^{*(n)}w_d^*(\omega) = f_d^*(\omega)$ is constant with respect to the starting network-coalition pair on each basin of attraction. In particular,

$$\lim_n p^{*(n)}w_d^*(\omega) = \int_\Omega w_d^*(\omega')d\lambda(\omega') = \int_{A_i} w_d^*(\omega')d\lambda_i(\omega') \text{ for all } \omega \in A_i.$$

By part (2) of Theorem 4, for any given invariant probability measure each player's average of time averages over the entire state space is equal to his state average over the entire state space with respect to the given measure.

6 Strategic Stability and Dynamic Consistency in Network and Coalition Formation

To begin, let $\sigma_D^{*\lambda}(\cdot) = (\sigma_d^{*\lambda}(\cdot|\cdot))_{d\in D}$ be the stationary correlated equilibrium with corresponding globally uniformly countable additive equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$. Also, let

$$\Omega = \left(\cup_{i=1}^N A_i\right) \cup T,$$

be the unique finite atomic decomposition generated by $p^*(\cdot|\cdot)$ with basins of attraction A_i and transient set T (where each A_i is an infinite atom) . Finally, let

$$\mathcal{E}^* = \{\lambda_i(\cdot)\}_{i=1}^N,$$

be the corresponding set of ergodic probability measures with $\lambda_i(A_i) = 1$ for all i.

Each player's strategy, $\sigma_d^{*\lambda}(\cdot|\cdot)$, is itself a Markov transition - an equilibrium Markov proposal transition - and governs the way in which player d tries to influence the process of network and coalition formation across time. The questions we wish to address in this section concern the relationships which exist between the invariance, ergodicity, and state space decomposition properties of Markov proposal transitions

$(\sigma_d^{*\lambda}(\cdot|\cdot))_{d\in D}$ and the invariance, ergodicity, and decomposition properties of the induced equilibrium Markov network-coalition transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$. When are they consistent in some sense and what implications does this have.

To save writing, we will refer to the equilibrium Markov proposal transitions, $(\sigma_d^{*\lambda}(\cdot|\cdot))_{d\in D}$, simply as the *proposal transitions*, and we will refer to the induced equilibrium Markov network-coalition transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$, as the *state transition.*

6.1 Strategic Stability and Dynamic Consistency

Given that the state space consists of network *and* coalition pairs and given the rules of network and coalition formation as represented via the set of feasible network-coalition pairs $\mathbb{G}$, the feasible set of coalitions $\mathcal{F} \subseteq P(D)$, and the player constraint correspondences,

$$\{\Phi_d(\cdot,\cdot)\}_{d\in D} = \{\Gamma_d(\cdot,\cdot) \times \Lambda_d(\cdot,\cdot)\}_{d\in D},$$

we are able to give formal definitions of strategic stability and dynamic consistency.

Definitions 5 (*Strategic Stability and Dynamic Consistency*)

(1) (*Strategic Stability*)

A set of network-coalition pairs $H \in B(\Omega)$ is strategically stable if in all states $(G,S) \in H$ each player $d \in D$ proposes states in H with probability 1, that is, if

$$\sigma_d^{*\lambda}(H|G,S) = 1 \text{ for all } (G,S) \in H.$$

(2) (Dynamic Consistency)

A strategically stable set of network-coalition pairs $H \in B(\Omega)$ is dynamically consistent if for in all states $(G,S) \in H$ nature chooses states in H with probability 1, that is, if

$$p^*(H|G,S) = 1 \text{ for all } (G,S) \in H.$$

Thus, a set of network-coalition pairs $H \in B(\Omega)$ is strategically stable if and only if

$$H \in \cap_{d\in D}\mathcal{L}_d^{*\lambda},$$

where $\mathcal{L}_d^{*\lambda}$ denotes the set of absorbing sets corresponding to player d's Markov proposal strategy $\sigma_d^{\lambda}(\cdot|\cdot)$, and H is dynamically consistent if and only if

$$H \in \cap_{d\in D}\mathcal{L}_d^{*\lambda} \cap \mathcal{L}^*$$

where recall $\mathcal{L}^*$ denotes the set of absorbing sets corresponding to the equilibrium Markov transition $p^*(\cdot|\cdot)$.

The following result characterizes dynamic strategic stability and dynamic consistency. The proof is straightforward.

Theorem 6 *(Dynamic Consistency and Invariance)*

Suppose assumptions [A-1], [A-2] and [A-3'] hold and let

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$.

If H *is dynamically consistent, then starting at any network-coalition pair contained in* H*, the network-coalition formation process will reach in finite time a basin* A_i*, that is, an infinite atom contained in* H*, and will remain there. Moreover, there exists a* p^**-invariant probability measure which assigns positive measure to* H*.*

Given the definition of an infinite atom and given that each basin A_i in the decomposition of the state space is an infinite atom, it must be true that any p^*-absorbing set contains one or more of the basins. Let us suppose then that dynamically consistent set $H \in B(\Omega)$ contains basins A_i and $A_{i'}$, and consider any p^*-invariant measure $\lambda(\cdot)$ such that $\lambda(H) = 1$. By part (2) of Theorem 4 above we have,

$$\lambda(H) = \sum_{i''}^{N} \lambda(A_{i''})\lambda_{i''}(H \cap A_{i''})$$

$$= \lambda(A_i)\lambda_i(H \cap A_i) + \lambda(A_{i'})\lambda_{i'}(H \cap A_{i'})$$

$$= \lambda(A_i) + \lambda(A_{i'}).$$

Thus, under any p^*-invariant measure $\lambda(\cdot)$ the measure of any absorbing set H is the sum of the probability masses the invariant measures $\lambda(\cdot)$ assigns to each basin (i.e., infinite atom) contained in H.

6.2 Dynamic Path dominance Core and Dynamic Pairwise Stability

One way to extend the definition of the path dominance core introduced in Page and Wooders (2007) to the dynamic setting considered here is as follows:

Definition 6 (*The Dynamic Path Dominance Core*)

A network $G^* \in \mathbb{G}$ *is in the dynamic path dominance core if the set of states* $\{G^*\} \times \mathcal{F} \in B(\Omega)$ *is dynamically consistent.*

We have the following characterization.

Theorem 7 *(The Dynamic Path Dominance Core and Invariance)*

Suppose assumptions [A-1], [A-2] and [A-3'] hold and let

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$.

If network $G^* \in \mathbb{G}$ *is in the dynamic path dominance core, that is, if* $\{G^*\} \times \mathcal{F}$ *is dynamically consistent, then starting at any network-coalition pair contained in* $\{G^*\} \times \mathcal{F}$*, the network-coalition formation process will reach in finite time a basin* A_i*, that is, an infinite atom contained in* $\{G^*\} \times \mathcal{F}$*, and will remain there. Moreover, there exists a* p^**-invariant probability measure which assigns positive measure to* $\{G^*\} \times \mathcal{F}$.

Because each basin is an infinite atom, we must conclude that if the dynamic path dominance core is nonempty, then it can contain no more than $|\mathcal{A}^{\infty *}|$ many networks. Moreover, each basin must be of the form

$$A_i = \{G^*\} \times \mathcal{C}_i,$$

where $\mathcal{C}_i \subseteq \mathcal{F}$ and $\mathcal{C}_i \cap \mathcal{C}_{i'} = \emptyset$ for all $i' \neq i$.

Note that if $p^*(\{G^*\} \times \mathcal{F}|G^*, S) = 1$ for all $S \in \mathcal{F}$, then because the law of motion

$$q(\cdot|(G, S), (G_D S_D))$$

is absolutely continuous with respect the probability measure $\mu = \nu \times \gamma$ for all $((G, S), (G_D, S_D)) \in Gr\Phi(\cdot)$, G^* must be an atom of the probability measure ν (not to be confused with an absorbing atom), that is,

$$G^* \in \mathbb{A}_\nu = \{G_{\alpha 1}, G_{\alpha 2}, \ldots\} = \{G_{\alpha k}\}_{k=1}^{\infty}.$$

Moreover, no network $G \in \mathbb{G} \backslash \mathbb{A}_\nu$, can be in the dynamic path dominance core.

To extend the definition of the pairwise stability introduced in Jackson and Wolinsky (1996) to the dynamic setting considered we begin by specializing the feasible set of coalitions to coalitions of size no greater than 2.

Definition 7 (*Dynamic Pairwise Stability*)

Suppose the feasible set of coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

(i.e., all feasible coalitions consist of at most two players). Then a network $G^* \in \mathbb{G}$ is dynamically pathwise stable if the set of states $\{G^*\} \times \mathcal{F}_2 \in B(\Omega)$ is dynamically consistent.

We have the following characterization

Theorem 8 *(Dynamic Pairwise Stability and Invariance)*

Suppose assumptions [A-1], [A-2] and [A-3'] hold and let

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^{*\lambda}(\cdot))$.

If network $G^* \in \mathbb{G}$ is dynamically pairwise stable, that is, if $\{G^*\} \times \mathcal{F}_2$ is dynamically consistent, then starting at any network-coalition pair contained in $\{G^*\} \times \mathcal{F}_2$, the network-coalition formation process will reach in finite time a basin A_i, that is, an infinite atom contained in $\{G^*\} \times \mathcal{F}_2$, and will remain there. Moreover, there exists a p^*-invariant probability measure which assigns positive measure to $\{G^*\} \times \mathcal{F}_2$.

Our conclusion that $\{G^*\} \times \mathcal{F}_2$ is contained in the support of some p^*-invariant measure is similar to the conclusion reached by Jackson and Watts (2002) for a stochastic process of network formation over a finite set of linking networks governed by Markov chain generated by myopic players. They reach their conclusion by considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain. This method is similar to a method introduced into games by Young (1993) which in turn is based on some very general perturbation methods found in Freidlin and Wentzell (1984). Here we have reached a similar conclusions without using perturbation methods.

7 Proof of Theorem 1: The Existence of Stationary Correlated Equilibrium

Proof. To begin let $\mathcal{V}$ be the set of all μ-equivalence classes of $B(\Omega)$-measurable functions, $v(\cdot) : \Omega \rightarrow [-M, M]$ called value functions. Because Ω is a compact metric space, the space of μ-equivalence classes of μ-integrable functions, $\mathcal{L}_1(\Omega, B(\Omega), \mu)$, is separable. As a consequence the set of value functions $\mathcal{V}$ is a compact and metrizable subset of $\mathcal{L}_\infty(\Omega, B(\Omega), \mu)$ for the weak star topology $\sigma(\mathcal{L}_\infty, \mathcal{L}_1)$. Letting

$$\mathcal{V}^m = \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{m:=|D| \text{ times}},$$

$\mathcal{V}^m$ equipped with the product topology $\sigma_m(\mathcal{L}_\infty, \mathcal{L}_1)$ is also compact and metrizable - and convex.

Given status quo state $\omega \in \Omega$, m-tuple of probability measure $\sigma = (\sigma_d) \in \Pi_{d\in D}\mathcal{P}(\Phi_d(\omega))$, and m-tuple of value functions $v = (v_d) \in \mathcal{V}^m$ define

$$u_d(\omega,\sigma)(v_d) := (1-\beta_d)r_d(\omega,\sigma) + \beta_d \int_\Omega v_d(\omega')dq(\omega'|\omega,\sigma).$$

The proof will proceed in 5 steps:

Step 1: Let

$$V(\omega,\sigma)(v) := \sum_d \left(u_d(\omega,(\sigma_d,\sigma_{-d}))(v_d) - \max_{\eta\in\mathcal{P}(\Phi_d(\omega))} u_d(\omega,(\eta,\sigma_{-d}))(v_d)\right),$$

and consider the correspondence $\omega \to N_v(\omega)$ where

$$N_v(\omega) := \{\sigma : V(\omega,\sigma)(v) = 0\}.$$

Note that $\sigma = (\sigma_d) \in N_v(\omega)$ if and only if for each player $d \in D$,

$$u_d(\omega,(\sigma_d,\sigma_{-d}))(v_d) \geq u_d(\omega,(\eta,\sigma_{-d}))(v_d) \text{ for all } \eta \in \mathcal{P}(\Phi_d(\omega)).$$

Thus, $\omega \to N_v(\omega)$ is the Nash correspondence. Given stochastic continuity assumption [A-3](ii) it follows from Delbaen's Lemma (1974) that

$$(\omega_d) \to \int_\Omega v_d(\omega')dq(\omega'|\omega,(\omega_d))$$

is continuous for any $v_d(\cdot) \in \mathcal{V}$. Thus, for $\omega \in \Omega$ and $v_d(\cdot) \in \mathcal{V}$

$$\sigma \to u_d(\omega,\sigma)(v_d) \text{ and } \sigma \to V(\omega,\sigma)(v)$$

are continuous on $\Pi_{d\in D}\mathcal{P}(\Phi_d(\omega))$ with respect to the compact and metrizable topology of weak convergence of probability measures. Thus, for all for $\omega \in \Omega$ and $v(\cdot) \in \mathcal{V}^m$, $N_v(\omega)$ is a nonempty, compact subset of $\Pi_{d\in D}\mathcal{P}(\Phi_d(\omega))$and by Theorem 6.4 in Himmelberg (1975) $N_v(\cdot)$ is measurable.

Step 2: Consider the induced payoff correspondence given by

$$P_v(\omega) := \{(U_d) \in R^m : (U_d) = (u_d(\omega,\sigma)(v_d)) \text{ for some } \sigma \in N_v(\omega)\}.$$

By Theorem 6.5 in Himmelberg (1975) the payoff correspondence $\omega \to P_v(\omega)$ is measurable with nonempty, compact values, and by Theorem 9.1 in Himmelberg (1975) the correspondence

$$\omega \to coP_v(\omega)$$

is measurable with nonempty, compact convex values.

Step 3: The Nowak-Raghavan Lemma.

Let $\Sigma(coP_v(\cdot))$ be the set of all μ-equivalence classes of measurable selectors of $\omega \to coP_v(\omega)$, $v \in \mathcal{V}^m$. The Nowak-Raghavan Lemma states that the payoff selection correspondence $v \to \Sigma(coP_v(\cdot))$ is upper semicontinuous with nonempty convex, weakly compact values. Convexity, weak compactness, and nonemptiness are straightforward. We need only prove upper semicontinuity. To this end, let $Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$ denote the graph of the payoff selection correspondence and let $\{(U^n(\cdot), v^n(\cdot))\}_n$ be a sequence in $Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$ converging weakly to $(U^*(\cdot), v^*(\cdot))$. In order to establish that the payoff selection correspondence is upper semicontinuous we must show that $(U^*(\cdot), v^*(\cdot)) \in Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$, that is, we must show that $U^*(\omega) \in coP_{v^*}(\omega)$ a.e. $[\mu]$.

The proof of this lemma proceeds in three steps:

First, given state $\omega \in \Omega$ and sequence $v^n(\cdot) \to v^*(\cdot)$, let $\{\sigma^n(\omega)\}_n$ be a sequence in $\Pi_{d\in D}\mathcal{P}(\Phi_d(\omega))$ such that $\sigma^n(\omega) \in N_{v^n}(\omega)$ for all n. Without loss of generality, suppose that $\sigma^n(\omega) \to \sigma^*(\omega) \in \Pi_{d\in D}\mathcal{P}(\Phi_d(\omega))$ with respect to the topology of weak convergence of probability measures. Then for all players $d \in D$,

$$u_d(\omega, \sigma^n(\omega))(v_d^n) \to u_d(\omega, \sigma^*(\omega))(v_d^*).$$

To see this, observe the following:

$$|u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^*)|$$

$$\leq \underbrace{|u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^n)|}_{A^n} + \underbrace{|u_d(\omega, \sigma^*(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^*)|}_{B^n}.$$

$$\underbrace{|u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^n)|}_{A^n}$$

$$\leq M\beta_d \left| \int_\Omega \int_{\Phi(\omega)} dq(\omega'|\omega, (\omega_d)) d\sigma^n((\omega_d)|\omega) \right.$$

$$\left. - \int_\Omega \int_{\Phi(\omega)} dq(\omega'|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega) \right|$$

$$= M\beta_d \left| \int_{\Phi(\omega)} q(\Omega|\omega, (\omega_d)) d\sigma^n((\omega_d)|\omega) \right.$$

$$\left. - \int_{\Phi(\omega)} q(\Omega|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega)) \right|.$$

$$\underbrace{|u_d(\omega, \sigma^*(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^*)|}_{B^n}$$

$$= \beta_d \left| \int_\Omega \int_{\Phi(\omega)} v_d^n(\omega') dq(\omega'|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega) \right.$$

$$\left. - \int_\Omega \int_{\Phi(\omega)} v_d^*(\omega') dq(\omega'|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega) \right|$$

$$= \beta_d \left| \int_{\Phi(\omega)} \int_\Omega v_d^n(\omega') dq(\omega'|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega) \right.$$

$$\left. - \int_{\Phi(\omega)} \int_\Omega v_d^*(\omega') dq(\omega'|\omega, (\omega_d)) d\sigma^*((\omega_d)|\omega) \right|.$$

By Delbaen's Lemma, $q(\Omega|\omega,(\omega_d))$ is continuous in (ω_d). Thus, since $\sigma^n(\cdot|\omega) \to \sigma^*(\cdot|\omega)$ with respect to weak convergence of probability measures,

$$M\beta_d \int_{\Phi(\omega)} q(\Omega|\omega,(\omega_d))d\sigma^n((\omega_d)|\omega) \to M\beta_d \int_{\Phi(\omega)} q(\Omega|\omega,(\omega_d))d\sigma^*((\omega_d)|\omega)),$$

so that $A^n \to 0$.

Next, given that the probability measures $q(\cdot|\omega,(\omega_d))$ are absolutely continuous with respect to probability measure μ, $v^n(\cdot) \to v^*(\cdot)$ weakly implies that for each (ω_d)

$$\int_\Omega v_d^n(\omega')dq(\omega'|\omega,(\omega_d)) \to \int_\Omega v_d^*(\omega')dq(\omega'|\omega,(\omega_d)).$$

In particular, we have for each $(\omega,(\omega_d))$

$$F^n(\omega,(\omega_d)) := \int_\Omega v_d^n(\omega')dq(\omega'|\omega,(\omega_d)) = \int_\Omega v_d^n(\omega')f(\omega'|\omega,(\omega_d))d\mu(\omega)$$

$$\to \int_\Omega v_d^*(\omega')f(\omega'|\omega,(\omega_d))d\mu(\omega) = \int_\Omega v_d^*(\omega')dq(\omega'|\omega,(\omega_d)) := F^*(\omega,(\omega_d)),$$

where $f(\omega'|\omega,(\omega_d))$ is the density of $q(\omega'|\omega,(\omega_d))$ with respect to μ. Thus, by the Dominated Convergence Theorem

$$\beta_d \int_{\Phi(\omega)} F^n(\omega,(\omega_d))d\sigma^*((\omega_d)|\omega) \to \beta_d \int_{\Phi(\omega)} F^*(\omega,(\omega_d))d\sigma^*((\omega_d)|\omega)),$$

so that $B^n \to 0$.

Therefore, we conclude that if $v^n(\cdot) \to v^*(\cdot)$ and $\sigma^n(\omega) \to \sigma^*(\omega)$, then for all players $d \in D$,

$$u_d(\omega,\sigma^n(\omega))(v_d^n) \to u_d(\omega,\sigma^*(\omega))(v_d^*).$$

Second, we have $U^n(\cdot) \to U^*(\cdot)$ weakly where for all n, $U^n(\cdot) \in \Sigma(coP_{v^n}(\cdot))$, and $v^n(\cdot) \to v^*(\cdot)$ weakly where for all n, $v^n(\cdot) \in \mathcal{V}^m$. By Proposition 1 in Page (1991), we can assume without loss of generality that for some μ null set N (i.e., $\mu(N)=0$)

$$\frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \to U(\omega) \text{ and } U(\omega) \in coLs\{U^n(\omega)\} \text{ for all } \omega \in \Omega \backslash N.$$

Here "*co*" denotes convex hull and $Ls\{U^n(\omega)\}$ is the set of limit point of the sequence $\{U^n(\omega)\}_n$. Now let $U^*(\cdot)$ be a measurable selector of $coLs\{U^n(\cdot)\}$ such that $U^*(\omega) = U(\omega)$ for all $\omega \in \Omega\backslash N$. Thus, $U^*(\omega) \in coLs\{U^n(\omega)\}$ for all $\omega \in \Omega$. By Theorem 8.2 in Wagner (1977) $U^*(\cdot)$ has a Caratheodory representation

$$U^*(\omega) = \sum_{i=0}^{m} \alpha^{*i}(\omega)U^{*i}(\omega)$$

where the R^m-valued functions $U^{*0}(\cdot), U^{*1}(\cdot), \ldots, U^{*m}(\cdot)$ are measurable selectors of $Ls\{U^n(\cdot)\}$ and the nonnegative functions $\alpha^{*0}(\cdot), \alpha^{*1}(\cdot), \ldots, \alpha^{*m}(\cdot)$ are measurable with $\sum_{i=0}^{m} \alpha^{*i}(\omega) = 1$ for all ω. Thus, for each i and each ω, $U^{n_k}(\omega) \to U^{*i}(\omega)$ for some subsequence $\{U^{n_k}(\omega)\}_k$.

Third, the proof that the payoff selection correspondence $v \to \Sigma(coP_v(\cdot))$ is upper semicontinuous, will be complete if we show that $U^{*i}(\omega) \in coP_{v^*}(\omega)$. To accomplish this, we need the following Lemma (*): If $U^n(\omega) \to U^{*i}(\omega)$ where $U^n(\omega) \in coP_{v^n}(\omega)$ for all n and $v^n(\cdot) \to v^*(\cdot)$ weakly, then $U^{*i}(\omega) \in coP_{v^*}(\omega)$.

Proof of Lemma (*): Again by Theorem 8.2 in Wagner (1977) each $U^n(\cdot)$ has a Caratheodory representation

$$U^n(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega) U^{ni}(\omega)$$

where each $U_i^n(\omega) \in P_{v^n}(\omega)$. Thus, for each n, there exists $\sigma_i^n(\omega) \in N_{v^n}(\omega)$ such that $U_i^n(\omega) = (u_d(\omega, \sigma_i^n(\omega))(v_d^n))$. Without loss of generality assume that

$$(\alpha^{n0}(\omega), \alpha^{n1}(\omega), \ldots, \alpha^{nm}(\omega)) \underset{n}{\to} (\alpha^{*0}(\omega), \alpha^{*1}(\omega), \ldots, \alpha^{*m}(\omega))$$
$$\text{and}$$
$$(\sigma_D^{n0}(\omega), \sigma_D^{n1}(\omega), \ldots, \sigma_D^{nm}(\omega)) \underset{n}{\to} (\sigma_D^{*0}(\omega), \sigma_D^{*1}(\omega), \ldots, \sigma_D^{*m}(\omega)).$$

Now we have

$$U^{ni}(\omega) = (u_d(\omega, \sigma_D^{ni}(\omega))(v_d^n)) \underset{n}{\to} (u_d(\omega, \sigma_D^{*i}(\omega))(v_d^*)) \in P_{v*}(\omega).$$

Thus,

$$U^n(\omega) = \textstyle\sum_{i=0}^{m} \alpha^{ni}(\omega) U^{ni}(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega)(u_d(\omega, \sigma_D^{ni}(\omega))(v_d^n))$$

$$\underset{n}{\to} \textstyle\sum_{i=0}^{m} \alpha^{*i}(\omega)(u_d(\omega, \sigma_D^{*i}(\omega))(v_d^*)) = U^{*i}(\omega) \in coP_{v^*}(\omega),$$

and we can conclude that

$$\text{for all } \omega,$$
$$U^*(\omega) = \textstyle\sum_{i=0}^{m} \alpha^{*i}(\omega) U^{*i}(\omega) \in coP_{v^*}(\omega),$$

completing the proof of the Nowak-Raghavan Lemma.

Step 4: Applying the Kakutani-Glicksberg Fixed Point Theorem (1952) to $v \to \Sigma(coP_v(\cdot))$ we obtain an m-tuple of value functions

$$v(\cdot) = (v_d(\cdot)) \in \mathcal{V}^m$$

such that

$$v(\omega) \in coP_v(\omega) \text{ for all } \omega \in \Omega \backslash N \text{ where } \mu(N) = 0.$$

Let $v^*(\cdot) = (v_d^*(\cdot)) \in \mathcal{V}^m$ be a measurable selection of $coP_v(\cdot)$ such that $v^*(\omega) = v(\omega)$ for all $\omega \in \Omega \backslash N$. Thus, $v^*(\omega) \in coP_v(\omega)$ for all $\omega \in \Omega$ and because $coP_v(\omega) = coP_{v^*}(\omega)$ for all $\omega \in \Omega$, we have $v^*(\omega) \in coP_{v^*}(\omega)$ for all $\omega \in \Omega$.

Step 5: Construct the solution to each player's dynamic programming problem: By Theorem 8.2 in Wagner (1977) $v^*(\cdot)$ has a Caratheodory representation

$$v^*(\omega) = \sum_{i=0}^{m} \lambda^{*i}(\omega) v^{*i}(\omega) \text{ for all } \omega$$

where for all $i = 0, 1, \ldots, m$, $v^{*i}(\cdot) \in \mathcal{V}^m$ and $v^{*i}(\omega) \in P_{v^*}(\omega)$ for all $\omega \in \Omega$. By the Measurable Implicit Function Theorem (Theorem 7.1 in Himmelberg (1975)), there exists for each $i = 0, 1, \ldots, m$, a measurable selection of $N_{v^*}(\cdot)$, that is, a measurable function

$$\omega \rightarrow \sigma_D^{*i}(\omega) \in \Pi_{d \in D} \mathcal{P}\left(\Phi_d(\omega)\right)$$

with $\sigma_D^{*i}(\omega) \in N_{v^*}(\omega)$ for all ω, such that for each player $d \in D$, $i = 0, 1, \ldots, m$, and $\omega \in \Omega$

$$v_d^{*i}(\omega) = u_d(\omega, \sigma_D^{*i}(\omega))(v_d^*) := (1 - \beta_d) r_d(\omega, \sigma_D^{*i}(\omega)) + \beta_d \int_\Omega v_d^*(\omega') dq(\omega' | \omega, \sigma_D^{*i}(\omega)).$$

Thus, for each player $d \in D$, and $\omega \in \Omega$

$$v_d^*(\omega) = \textstyle\sum_{i=0}^m \lambda^{*i}(\omega) v_d^{*i}(\omega)$$

$$= (1 - \beta_d) r_d(\omega, \textstyle\sum_{i=0}^m \lambda^{*i}(\omega) \sigma_D^{*i}(\omega)) + \beta_d \int_\Omega v_d^*(\omega') dq(\omega' | \omega, \sum_{i=0}^m \lambda^{*i}(\omega) \sigma_D^{*i}(\omega))$$

For $d \in D$, let $w_d^*(\cdot) := \frac{v_d^*(\cdot)}{1 - \beta_d}$. Substituting, we have for all $\omega \in \Omega$

$$w_d^*(\omega) = r_d(\omega, \sigma_D^{*\lambda}(\omega)) + \beta_d \int_\Omega w_d^*(\omega') dq(\omega' | \omega, \sigma_D^{*\lambda}(\omega)). \ (**)$$

where $\sigma_D^{*\lambda}(\omega) = \sum_{i=0}^m \lambda^{*i}(\omega) \sigma_D^{*i}(\omega) \in coN_{w^*}(\omega)$ for all ω.

By classical results on discounted dynamic programming (e.g., Blackwell (1965)), we conclude from (**) that (i) for all players $d \in D$ and all starting states $\omega \in \Omega$

$$w_d^*(\omega) = E_d(\sigma_D^{*\lambda})(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(\sigma_D^{*\lambda})(\omega),$$

and therefore that (ii) for all players $d \in D$ and all starting states $\omega \in \Omega$

$$E_d(\sigma_d^{*\lambda}, \sigma_{-d}^{*\lambda})(\omega) \geq E_d(\pi_d, \sigma_{-d}^{*\lambda})(\omega) \text{ for all } \pi_d \in \Pi_d^\infty.$$

■

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